ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 23 | Modern Digital Communications

Final Exam Solutions | December 19, 2018

PROBLEM 1. (Paper and Pencil)

Note: the message in this problem is that we can do the up-conversion from baseband to passband by an appropriate choice of the interpolation filter if the target center frequency f_c is an integer multiple of $\frac{1}{T_s}$. If the center frequency f_d is an arbitrary one, then an additional pre-processing is needed.

1. $T_s < \frac{1}{B}$.

2. The sampling theorem tells us that we can reconstruct $b(t)$ as follows:

$$b(t) = \sum_i b(iT_s) \operatorname{sinc}\left(\frac{t - iT_s}{T_s}\right).$$

Hence,

$$b(t) = \sum_i b(iT_s) h_0(t - iT_s).$$

3. $p_{\mathcal{F}}(f) = b_{\mathcal{F}}(f - f_c)$.

4. $p(nT_s) = b(nT_s)e^{j2\pi f_c nT_s}$. Hence, if we wish to have $p(nT_s) = b(nT_s)$ for every integer n, we should have that $f_c T_s = k$ for some integer k. So we should have $f_c = \frac{k}{T_s}$.

5.

$$\begin{aligned}
\sum_i b(iT_s) h_c(t - iT_s) &= \sum_i b(iT_s) e^{j2\pi f_c(t - iT_s)} \operatorname{sinc}\left(\frac{t - iT_s}{T_s}\right) \\
&= e^{j2\pi f_c t} \sum_i b(iT_s) \operatorname{sinc}\left(\frac{t - iT_s}{T_s}\right) \\
&= b(t) e^{j2\pi f_c t} \\
&= p(t).
\end{aligned}$$

6.

$$\begin{aligned}
\sum_i b(iT_s) q_i h_d(t - iT_s) &= \sum_i b(iT_s) q_i e^{j2\pi f_d(t - iT_s)} \operatorname{sinc}\left(\frac{t - iT_s}{T_s}\right) \\
&= e^{j2\pi f_d t} \sum_i b(iT_s) q_i e^{-j2\pi (f_c + \Delta) iT_s} \operatorname{sinc}\left(\frac{t - iT_s}{T_s}\right) \\
&= e^{j2\pi f_d t} \sum_i b(iT_s) q_i e^{-j2\pi \Delta iT_s} \operatorname{sinc}\left(\frac{t - iT_s}{T_s}\right).
\end{aligned}$$

We immediately see that if $q_i = e^{j2\pi \Delta iT_s}$ the above becomes

$$e^{j2\pi f_d t} \sum_i b(iT_s) \operatorname{sinc}\left(\frac{t - iT_s}{T_s}\right) = b(t) e^{j2\pi f_d t} = q(t).$$

Problem 2. (Paper and Pencil)

1. C is an $N+L-1 \times L$ matrix:

$$C = \begin{pmatrix} x_0 & 0 & \dots & 0 \\ x_1 & x_0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ x_{N-1} & x_{N-2} & \dots & x_{N-L} \\ 0 & x_{N-1} & \dots & x_{N-L+1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & x_{N-1} \end{pmatrix}.$$

2. We are seeking for the $\hat{\boldsymbol{h}}$ which minimizes

$$\|\boldsymbol{y} - C\boldsymbol{h}\|^2$$

   over all $\boldsymbol{h}$.

3. $C\hat{\boldsymbol{h}}$ is an element of the inner-product space spanned by the columns of C. So $\mathcal{V} = \mathrm{span}\{C_1, \dots, C_L\}$ where C_i is the ith column of C, $i = 1, 2, \dots, L$.

4. We are seeking the vector $\hat{\boldsymbol{y}} \in \mathcal{V}$ which minimizes

$$\|\boldsymbol{y} - \hat{\boldsymbol{y}}\|^2.$$

   The projection theorem tells us that $\hat{\boldsymbol{y}}$ is the projection of $\boldsymbol{y}$ into $\mathcal{V}$. It has the property that the error vector $\boldsymbol{y} - \hat{\boldsymbol{y}}$ is orthogonal to every element of $\mathcal{V}$. In particular, it is orthogonal to the columns of C. Hence

$$\langle \boldsymbol{y} - C\hat{\boldsymbol{h}}, C_i \rangle = 0, \quad i = 1, 2, \dots, L.$$

   Hence

$$\langle \boldsymbol{y}, C_i \rangle = \langle C\hat{\boldsymbol{h}}, C_i \rangle, \quad i = 1, 2, \dots, L.$$

   Equivalently,

$$C_i^\dagger \boldsymbol{y} = C_i^\dagger C\hat{\boldsymbol{h}} \quad i = 1, 2, \dots, L.$$

   In matrix form

$$C^\dagger \boldsymbol{y} = C^\dagger C\hat{\boldsymbol{h}}.$$

5. Solving for $\hat{\boldsymbol{h}}$ yields the least-squares approximation of $\boldsymbol{h}$:

$$\hat{\boldsymbol{h}} = \left(C^\dagger C\right)^{-1} C^\dagger \boldsymbol{y}.$$