ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 14 Modern Digital Communications
Solutions to Assignment 9 November 27, 2024

For the first 3 exercises, please see the provided MATLAB/Python code, which is self-explanatory.

SOLUTION 4.

1. By the linearity of convolution, we have
$$r(t) = s(t) \star h(t) + N(t) = \sum_k a[k](p \star h)(t - kT) + N(t).$$

2. Let $\tilde{q}(t) \coloneqq q^*(-t)$ for the sake of brevity. We have
$$y[k] = y(kT) = (s \star h \star \tilde{q})(kT) + \int N(\alpha)\tilde{q}(kT - \alpha)d\alpha$$
Since $N(t)$ is a zero-mean white Gaussian process,
$$z[k] \coloneqq \int N(\alpha)\tilde{q}(kT - \alpha)d\alpha = \int N(\alpha)q^*(\alpha - kT)d\alpha$$
is a collection of zero-mean jointly Gaussian random variables with covariance (see the PDC notes)
$$E\big[z[k]z[j]^*\big] = N_0\langle q(t - jT), q(t - kT)\rangle.$$
Finally
$$(s \star h \star \tilde{q})(t) = \sum_j a[j](p \star h \star \tilde{q})(t - jT).$$
Therefore,
$$(s \star h \star \tilde{q})(kT) = \sum_j a[j](p \star h \star \tilde{q})(kT - jT) = \sum_n a[k - n]\underbrace{(p \star h \star \tilde{q})(nT)}_{=:h[n]}$$
Putting everything together we get
$$y[k] = \sum_n a[k - n]h[n] + z[k],$$
where the symbol-level equivalent channel is
$$h[n] = p(t) \star h(t) \star q^*(-t)\Big|_{t=nT}.$$

3. From our results of part 2, we know that if $q(t)$ is a Nyquist pulse, i.e., $\langle q(t - kT), q(t - jT)\rangle = \mathbb{1}\{k = j\}$ the noise process $z[k]$ will be white (i.e., an i.i.d. Gaussian sequence).

4. From PDC we know that sufficient statistics for decisions are obtained by projecting the received signal onto the space spanned by the set of signals. Therefore, if $\{q(t - kT)\}_{k\in\mathbb{Z}}$ forms a basis for the signal space spanned by $\{(p \star h)(t - kT)\}_{k\in\mathbb{Z}}$, the observables $y[k]$ will be sufficient statistics for decisions. To see this, suppose that $\psi_1(t), \dots, \psi_n(t)$ is an orthonormal basis, and $v_1(t), \dots, v_n(t)$ is another basis, not necessarily orthonormal. Let $r_i = \langle r(t), \psi_i(t)\rangle$ be the projections that form a sufficient statistic. Let $t_i = \langle r(t), v_i(t)\rangle$ be the projections on the second basis. We want to show that we can recover $r_1, \dots, r_n$ from $t_1, \dots, t_n$.
$$\begin{aligned} r_i = \langle r(t), \psi_i(t)\rangle &\overset{(i)}{=} \langle r(t), \sum_j \alpha_{ij}v_j(t)\rangle \\ &= \sum_j \langle r(t), v_j(t)\rangle\alpha_{ij}^* \\ &= \sum_j t_j\alpha_{ij}^* , \end{aligned}$$
where in (i) we used the fact that $\psi_i(t)$ can be written as a linear combination of $v_1(t), \dots, v_n(t)$.

5. With this choice of $h(t)$,

$$p(t) \star h(t) \star q^*(-t) = f(t) \star h(t) = \sum_{l=0}^{M-1} \alpha_l f(t - \tau_l).$$

Consequently

$$p(t) \star h(t) \star q^*(-t)\Big|_{t=nT} = \sum_{l=0}^{M-1} \alpha_l f(nT - \tau_l).$$