Modern Digital Communications:
A Hands-On Approach

GPS: From Pseudoranges and Ephemerides to Positioning

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Goal

The goal of this module is to summarize how to compute the receiver position given the data (ephemeris) and the pseudorange for at least 4 satellites.

Model Versus Reality

So far we have assumed an ideal model. In reality (see e.g., Misra and Enge, p. 75),

- The earth is not round, there is a gravitational field due to the sun and the moon, and there is solar radiation pressure. All these perturb the satellite orbit.
- The earth's rotational axis is subject to precession (periodic oscillation with a period of 26,000 years due to the torque created by the sun and the moon on the non-homogeneous earth) and nutation (period 18.6 years).
- The earth's angular speed is not uniform.
- The earth moves with respect to the earth's rotational axis. This is called *polar motion*. (About 15m per year.)

Because of the above effects, some of the "constants" actually have correction terms that depend on time. Here they are:

$$\Omega_{er} = \Omega_0 - \dot{\Omega}_e t \;\Rightarrow\; \Omega_0 + \dot{\Omega}(t - t_{oe}) - \dot{\Omega}_e t.$$

You may think of $\Omega_0 + \dot{\Omega}(t - t_{oe})$ as a Taylor series expansion of Ω_0 around the time t_{oe} called *reference time of the ephemeris*.

The parameters Ω_0, $\dot{\Omega}$, and t_{oe} are sent by the satellite as part of the ephemeris data.

$$n \;\Rightarrow\; n + \Delta n = \sqrt{\frac{\mu}{a^3}} + \Delta n. \qquad \text{(Recall that } M = M_0 + n(t - t_{oe})\text{)}$$

$$\phi^{(k)} = \nu^{(k)} + \omega \;\Longrightarrow\; \nu^{(k)} + \omega + c_{\omega s} \sin\left(2(\nu^{(k)} + \omega)\right) + c_{\omega c} \cos\left(2(\nu^{(k)} + \omega)\right).$$

$$r^{(k)} \;\Rightarrow\; a(1 - e\cos E^{(k)}) + c_{rs} \sin\left(2(\nu^{(k)} + \omega)\right) + c_{rc} \cos\left(2(\nu^{(k)} + \omega)\right) \;.$$

$$i^{(k)} \;\Rightarrow\; i_0 + c_{is} \sin\left(2(\nu^{(k)} + \omega)\right) + c_{ic} \cos\left(2(\nu^{(k)} + \omega)\right) + IDOT(t - t_{oe}).$$

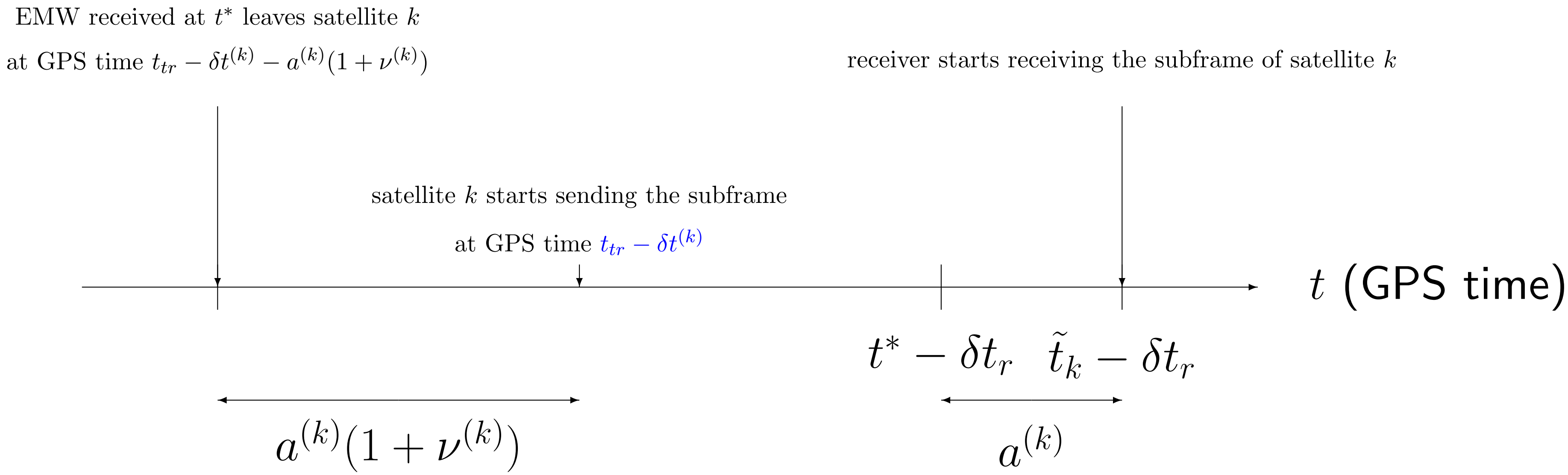
TOW

- The *time of the week* TOW is a parameter sent by the satellite in each subframe.

- It is the GPS time at which the next subframe is supposed to start.

- Hence, $t_{tr} = \mathrm{TOW} - 6$ refers to the GPS time at which the current subframe was supposed to start.

- In reality, due to the satellite clock error $\delta t^{(k)}$, the current subframe of satellite k started at GPS time $t_{tr} - \delta t^{(k)}$.



EMW received at t^* leaves satellite k
at GPS time $t_{tr} - \delta t^{(k)} - a^{(k)}(1+\nu^{(k)})$
receiver starts receiving the subframe of satellite k
satellite k starts sending the subframe
at GPS time $t_{tr} - \delta t^{(k)}$
t (GPS time)
$t^* - \delta t_r$
$\tilde{t}_k - \delta t_r$
$a^{(k)}(1+\nu^{(k)})$
$a^{(k)}$

Getting The Position in 8 Steps

The following steps lead to the receiver position. (For more information see Table 20-IV (pages 97,98) of the official GPS document that you can find on the course web page.)

Step 1: Find the ephemerides of the "visible" satellites and the pseudoranges at some receiver time t^* (the same for all satellites). (We have this part from the previous assignment.)

Step 2: Write `calcE(ephdata,t)`.

This function returns the eccentric anomaly $E^{(k)}$ at GPS time t.

Recall that

$$M^{(k)} = E^{(k)} - e \sin E^{(k)},$$

where

- $M^{(k)} = M_0 + n(t - t_{oe})$ is the mean anomaly.
- $n = \sqrt{\frac{\mu}{a^3}} + \Delta n$ is the corrected mean motion,
- $\mu = 3.986005 \times 10^{14}$ $\mathrm{m}^3/\mathrm{sec}^2$ is the earth's gravitational constant
- t_{oe} is the reference time of the ephemeris,
- e, M_0, Δn

are all contained in `ephdata`.

We find the solution iteratively: Dropping the superscript, we start with

$$E_0 = M$$

and for $i = 0, 1, \ldots$ we compute

$$E_{i+1} = M + e \sin E_i$$

until the change is negligible.

Step 3: Write `calcDeltaT(ephdata,E_k,t)`.

This function returns $\delta t^{(k)}$ at any desired GPS time t, where

$$\delta t^{(k)} = a_{f0} + a_{f1}(t - t_{oc}) + a_{f2}(t - t_{oc})^2 + \Delta t,$$

where

- $\Delta t = Fe\sqrt{a}\sin E^{(k)}$ is the relativistic term and
- $F = \frac{-2\sqrt{\mu}}{c^2} = -4.442807633 \times 10^{-10}$ sec/m$^{1/2}$ is a constant,
- t_{oc} is a clock data reference time,
- and a_{f0}, a_{f1}, a_{f2} are the coefficients of the Taylor expansion.

All parameters are contained in `ephdata`.

Step 4: Write `satpos(ephdata,t)`.

This function returns $p^{(k)}(t,t)$ via the following steps:

- $\nu^{(k)} = \tan^{-1}\left(\frac{\sqrt{1-e^2}\sin E^{(k)}}{\cos E^{(k)}-e}\right)$ is the true anomaly

- $r^{(k)} = a(1 - e\cos E^{(k)}) + c_{rs}\sin\left(2(\nu^{(k)}+\omega)\right) + c_{rc}\cos\left(2(\nu^{(k)}+\omega)\right)$ is the radius with correction terms

The above gives us the satellite's position in polar coordinates with respect to the reference system matched to its orbit.

Next we convert to cartesian coordinates and rotate the coordinate system around the z axis so as to have x_{OP} on the equatorial plane.

- $\phi^{(k)} = \nu^{(k)} + \omega + c_{\omega s} \sin\left(2(\nu^{(k)} + \omega)\right) + c_{\omega c} \cos\left(2(\nu^{(k)} + \omega)\right)$ is the true anomaly plus the argument of perigee with correction terms,

- $p_{OP}^{(k)} = \begin{pmatrix} r^{(k)} \cos\phi^{(k)} \\ r^{(k)} \sin\phi^{(k)} \\ 0 \end{pmatrix}$

Next we rotate around the x axis by $-i^{(k)}$ radiants and around the z axis by $-\Omega_{er}^{(k)}$:

- $i^{(k)} = i_0 + c_{is} \sin\left(2(\nu^{(k)} + \omega)\right) + c_{ic} \cos\left(2(\nu^{(k)} + \omega)\right) + IDOT(t - t_{oe})$,

- $\Omega_{er}^{(k)} = \Omega_0 + \dot{\Omega}(t - t_{oe}) - \dot{\Omega}_e t$, where $\dot{\Omega}_e = 7.2921151467 \times 10^{-5}$,

- $p(t,t) = p_G = R_3(-\Omega_{er}^{(k)}) R_1(-i^{(k)}) p_{OP}^{(k)}$.

All the parameters are contained in `ephdata`.

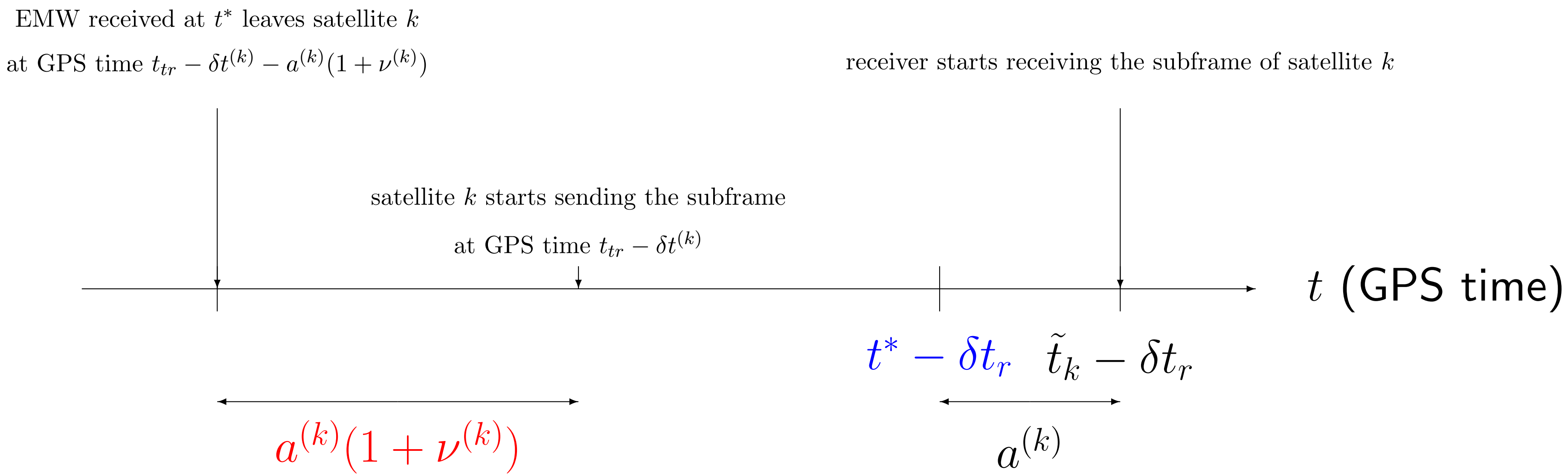
With $p^{(k)}(t,t)$, we can determine $p^{(k)}(t,\xi)$ for any t and ξ.

Step 5: With reference to the figure below, determine

$$p^{(k)} := p^{(k)}\left(t_{tr} - \delta t^{(k)} - a^{(k)}(1+\nu^{(k)}), t_{tr}\right).$$

This is the position of the satellite when it sends the signal received at (receiver time) t^*, expressed in ECEF(t_{tr}).

The specific value of ξ ($\xi = t_{tr}$ in our case) is not important but it is important that we choose the same ξ for all satellites, so that all positions are expressed in a common reference frame.



Recall that $t^* - \delta t_r$ is the GPS time for which we want to determine the receiver position, and

$$p^{(k)} := p^{(k)}\left(t_{tr} - \delta t^{(k)} - a^{(k)}(1 + \nu^{(k)}), t_{tr}\right)$$

is the position of the satellite when it sent the signal received at GPS time $t^* - \delta t_r$.

Notice that the $\delta t^{(k)}$ on the right of $t := t_{tr} - \delta t^{(k)} - a^{(k)}(1 + \nu^{(k)})$ is the one that holds at time t. There is a 'chicken-egg' problem.

We determine t iteratively as follows. Let `t` be $t_{tr} - a^{(k)}(1 + \nu^{(k)})$, where t_{tr} is sent by the satellite and $a^{(k)}(1 + \nu^{(k)})$ is measured by the receiver.

```
Delta = 0;
while true % infinite loop
    t=t-Delta
    E=calcE(ephdata, t)
    Delta_new = calcDeltaT(ephdata, E, t)
    if (abs(Delta_new - Delta)<=1e-10)
    break;
    end
    Delta=Delta_new;
end
```

When the loop breaks, `t-Delta_new` equals $t_{tr} - \delta t^{(k)} - a^{(k)}(1 + \nu^{(k)})$.

The above is a disguised form of iteration to find the fixed point of a function.

Step 6: If we could, we would write 3 or more equations such as

$$\|p^{(k)} - \tilde{p}\| = c\tau_f^k,$$

and solve for $\tilde{p}$.

We don't have $c\tau_f^k$ but we have $\rho_c^{(k)}$, where where

$$c\tau_f^k = c\left(t^* - \delta t_r - t_{tr} + \delta t^{(k)} + (a^{(k)}(1 + \nu^{(k)})\right) \text{ is the range (unknown)},$$
$$\rho_c^{(k)} = c\left(\delta t^{(k)} + a^{(k)}(1 + \nu^{(k)}\right) \text{ is the corrected pseudorange (known)}.$$

By letting $b = c\tau_f^k - \rho_c^{(k)}$ (unknown), we write the equations

$$\|p^{(k)} - \tilde{p}\| = b + \rho_c^{(k)},$$

for at least 4 satellites and solve for $\tilde{p}$ and b.

Step 7: Notice that

$$\begin{aligned} b &= c\tau_f^k - \rho_c^{(k)} \\ &= c(t^* - \delta t_r - t_{tr}) \end{aligned}$$

can be rewritten as

$$t^* - \delta t_r = \frac{b}{c} + t_{tr}.$$

The RHS is known, and the LHS is the GPS time that corresponds to the receiver time t^*. From this moment on, the receiver knows the GPS time.

Recall that $\tilde{p}$ is the receiver position at GPS time $t^* - \delta t_r$, with respect to the ECEF(t_{tr}) coordinate system.

What we want is $p := p\,(t^* - \delta t_r, t^* - \delta t_r)$

Reusing the fact that $t^* - \delta t_r = \frac{b}{c} + t_{tr}$ we obtain

$$
\begin{aligned}
p &= p\,(t^* - \delta t_r, t^* - \delta t_r) \\
&= p\left(t^* - \delta t_r, \frac{b}{c} + t_{tr}\right) \\
&= R_3\left(\dot{\Omega}_e \frac{b}{c}\right) p\,(t^* - \delta t_r, t_{tr}) \\
&= R_3\left(\dot{\Omega}_e \frac{b}{c}\right) \tilde{p}.
\end{aligned}
$$

Step 8: We convert $p = (x, y, z)$ to coordinates expressed in terms of latitude l, longitude L, and elevation h.

For comparison, and since some of the terms are in common, it is convenient to first write down the polar coordinates (l, L_p, r):

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ l &= \tan^{-1}\left(\frac{y}{x}\right) \\ L_p &= \tan^{-1}\left(z/\sqrt{x^2 + y^2}\right). \end{aligned}$$

If the earth were a sphere, then we just would write $L = L_p$ and $h = r - r_0$ with r_0 being the radius of the earth.

However, the earth is rather a squashed sphere (squeezed at the poles).

The GPS standard uses the WGS84 geodetic coordinates that may be obtained via the following rather accurate approximation (with $e = 0.00335281066474$ and $a = 6378137$)

$$
\begin{aligned}
l &= \tan^{-1}\left(\frac{y}{x}\right)\\
L &\approx L_p + e\sin(2L)\\
h &\approx r - r_0,\ \text{where } r_0 \approx a(1 - e\sin^2 L).
\end{aligned}
$$

Once you have obtained the coordinates, you may check the location by inserting the latitude and longitude in `http://maps.google.com`. Here is the result: