

PROBLEM 1. 18 points (Paper and Pencil)

1. By the linearity of convolution, we have

$$r(t) = s(t) \star h(t) + N(t) = \sum_k s[k](p \star h)(t - kT) + N(t) = \sum_k s[k]g(t - kT) + N(t),$$

where $g(t) = (p \star h)(t)$.

2. Again, by the linearity of the convolution, we have

$$y(t) = r(t) \star q(t) = \sum_k s[k](p \star h \star q)(t - kT) + \int N(\alpha)q(t - \alpha)d\alpha.$$

Hence,

$$y(t) = \sum_k s[k]f(t - kT) + \int N(\alpha)q(t - \alpha)d\alpha,$$

where $f(t) = (p \star h \star q)(t)$.

- 3.

$$y[k] = y(kT) = \sum_j s[j]f(kT - jT) + \int N(\alpha)q(kT - \alpha)d\alpha.$$

Hence,

$$y[k] = \sum_n \underbrace{(p \star h \star q)(nT)}_{=:v[n]} \underbrace{s[k - n]}_{=:b_{k-n}} + z[k],$$

where $z[k] = \int N(\alpha)q(kT - \alpha)d\alpha$.

4. Since $N(t)$ is a zero-mean white Gaussian process, $z[k]$ is a collection of zero-mean jointly Gaussian random variables with covariance

$$E[z[k]z[j]^*] = N_0 \langle q(kT - t), q(jT - t) \rangle.$$

5. Under the new assumptions,

$$v[n] = (p \star q)(nT) = \int p(\alpha)q(nT - \alpha)d\alpha = \int p(\alpha)p^*(\alpha - nT)d\alpha = \langle p(\alpha), p(\alpha - nT) \rangle = \delta_n.$$

Hence, $y[k] = s[k] + z[k]$, that is, there is no ISI (Inter-Symbol Interference).

$$E[z[k]z[j]^*] = N_0 \langle q(kT - t), q(jT - t) \rangle = N_0 \langle p(t - jT), p(t - kT) \rangle = N_0 \delta_{j-k}.$$

Hence, the noise process will have independent components.

PROBLEM 3. 21 points (Paper and Pencil + MATLAB/Python)

1. We have shown in class that the LS estimate of \mathbf{a} is $\hat{\mathbf{a}}_{\text{LS}} = (D^\dagger D)^{-1} D^\dagger \mathbf{y}$. Since D is a non-singular diagonal matrix, $(D^\dagger D)^{-1} = D^{-1} (D^\dagger)^{-1}$, so the previous expression simplifies to

$$\hat{\mathbf{a}}_{\text{LS}} = D^{-1} \mathbf{y}.$$

2. We have shown in class that the LMMSE estimate of \mathbf{a} is $\hat{\mathbf{a}}_{\text{LMMSE}} = K_{\mathbf{a}\mathbf{y}} K_{\mathbf{y}}^{-1} \mathbf{y}$. We can easily compute the covariance matrices:

$$K_{\mathbf{a}\mathbf{y}} = E[\mathbf{a}\mathbf{y}^\dagger] = E[\mathbf{a}(D\mathbf{a} + \mathbf{z})^\dagger] = E[\mathbf{a}\mathbf{a}^\dagger] D^\dagger = D^\dagger,$$

and

$$K_{\mathbf{y}} = E[\mathbf{y}\mathbf{y}^\dagger] = E[(D\mathbf{a} + \mathbf{z})(D\mathbf{a} + \mathbf{z})^\dagger] = DE[\mathbf{a}\mathbf{a}^\dagger] D^\dagger + E[\mathbf{z}\mathbf{z}^\dagger] = DD^\dagger + \sigma^2 I_N.$$

Hence,

$$\hat{\mathbf{a}}_{\text{LMMSE}} = D^\dagger (DD^\dagger + \sigma^2 I_N)^{-1} \mathbf{y}.$$