Homework 10

Exercise 1.*

Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. non-negative random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that there exists $0 < a < b < +\infty$ with $a < X_n(\omega) \leq b$ for all $n \geq 1$ and $\omega \in \Omega$. Let also $(Y_n,\, n \geq 1)$ be the sequence defined as

$$Y_n = \left(\prod_{j=1}^{n} X_j \right)^{1/n}, \quad n \geq 1$$

a) Show that there exists a constant $\mu > 0$ such that $Y_n \underset{n\to\infty}{\to} \mu$ almost surely.

b) Compute the value of μ in the case where $\mathbb{P}(\{X_1 = a\}) = \mathbb{P}(\{X_1 = b\}) = \frac{1}{2}$ and $a, b > 0$.

c) In this case, look for a good upper bound on $\mathbb{P}(\{Y_n > t\})$ for $n \geq 1$ fixed and $t > \mu$.

Exercise 2. (The Birthday Problem)
Let $(X_n, n \geq 1)$ be a sequence of i.i.d. random variables, each uniform on $\{1, \ldots, N\}$. Let also

$$T_N = \min\{n \geq 1 \; : \; X_n = X_m \text{ for some } m < n\}$$

(notice that whatever happens, $T_N \in \{2, \ldots, N+1\}$).

a) Show that

$$\mathbb{P}\left(\left\{ \frac{T_N}{\sqrt{N}} \leq t \right\} \right) \underset{N\to\infty}{\to} 1 - e^{-t^2/2}, \quad \forall t \geq 0$$

Remarks:
- Approximations are allowed here!

- Please observe that the limit distribution is *not* the Gaussian distribution!

b) *Numerical application:* Use this to obtain a rough estimate of $\mathbb{P}(\{T_{365} \leq 22\})$ and $\mathbb{P}(\{T_{365} \leq 50\})$ (i.e., what is the probability that among 22 / 50 people, at least two share the same birthday?)

Exercise 3. (Application of Lindeberg's Principle to Non I.I.D. Random Variables)
Let $(\sigma_n,\, n \geq 0)$ be a sequence of (strictly) positive numbers and $(X_n,\, n \geq 1)$ be a sequence of independent random variables such that $\mathbb{E}(X_n) = 0$, $\mathrm{Var}(X_n) = \sigma_n^2$ and $\mathbb{E}(|X_n|^3) \leq K\,\sigma_n^3$ for every $n \geq 1$ (note that the constant K is uniform over all values of n).

For $n \geq 1$, define also $V_n = \mathrm{Var}(X_1 + \ldots + X_n) = \sigma_1^2 + \ldots + \sigma_n^2$.

a) Using Lemma 9.12 in the lecture notes, find a sufficient condition on the sequence $(\sigma_n,\, n \geq 1)$ guaranteeing that

$$\frac{1}{\sqrt{V_n}} (X_1 + \ldots + X_n) \underset{n\to\infty}{\overset{d}{\to}} Z \sim \mathcal{N}(0,1)$$

Note: From the course, you already know a sufficient condition: $\sigma_n = 1$ for all $n \geq 1$, but this is too strong! The aim here is to find a sufficient condition which is most general possible.

b) Which of the following sequences $(\sigma_n,\, n \geq 1)$ satisfy the condition you have found in a)?

b1) $\sigma_n = n$ b2) $\sigma_n = \frac{1}{n}$ b3) $\sigma_n = 2^n$

Hint: The following might be useful: $\sum_{j=1}^{n} j^\alpha = \begin{cases} \Theta(n^{\alpha+1}) & \text{if } \alpha > -1 \\ \Theta(\log(n)) & \text{if } \alpha = -1 \\ \Theta(1) & \text{if } \alpha < -1 \end{cases}$

Exercise 4. Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d and square integrable random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\mathbb{E}(X_1) = 1$ and $\mathrm{Var}(X_1) = \sigma^2$. Let $S_n = \sum_{i=1}^n X_i$. Then, show that

$$\frac{2}{\sigma}(\sqrt{S_n} - \sqrt{n}) \xrightarrow{d} X$$

where X is a standard Gaussian random variable i.e., $\mathcal{N}(0,1)$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (For simplicity, assume also that $\mathbb{E}(|X_1|^3) < +\infty$).