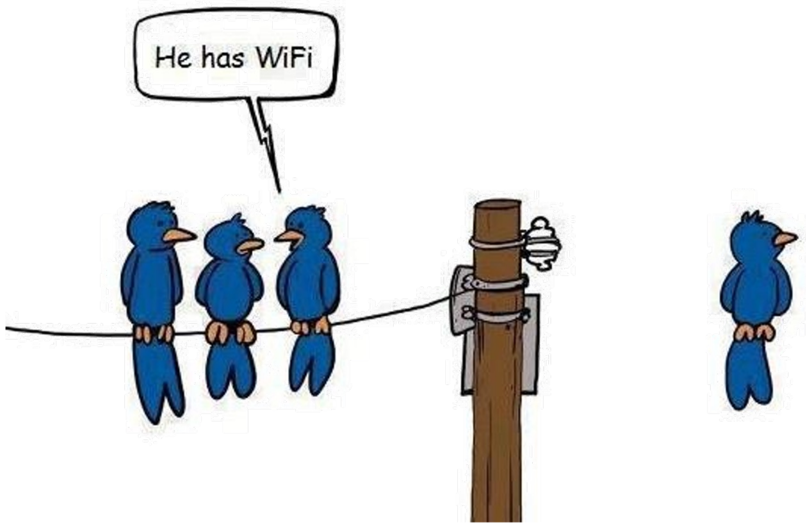
COM-405: Mobile Networks

Lecture 3.1: Modulation I

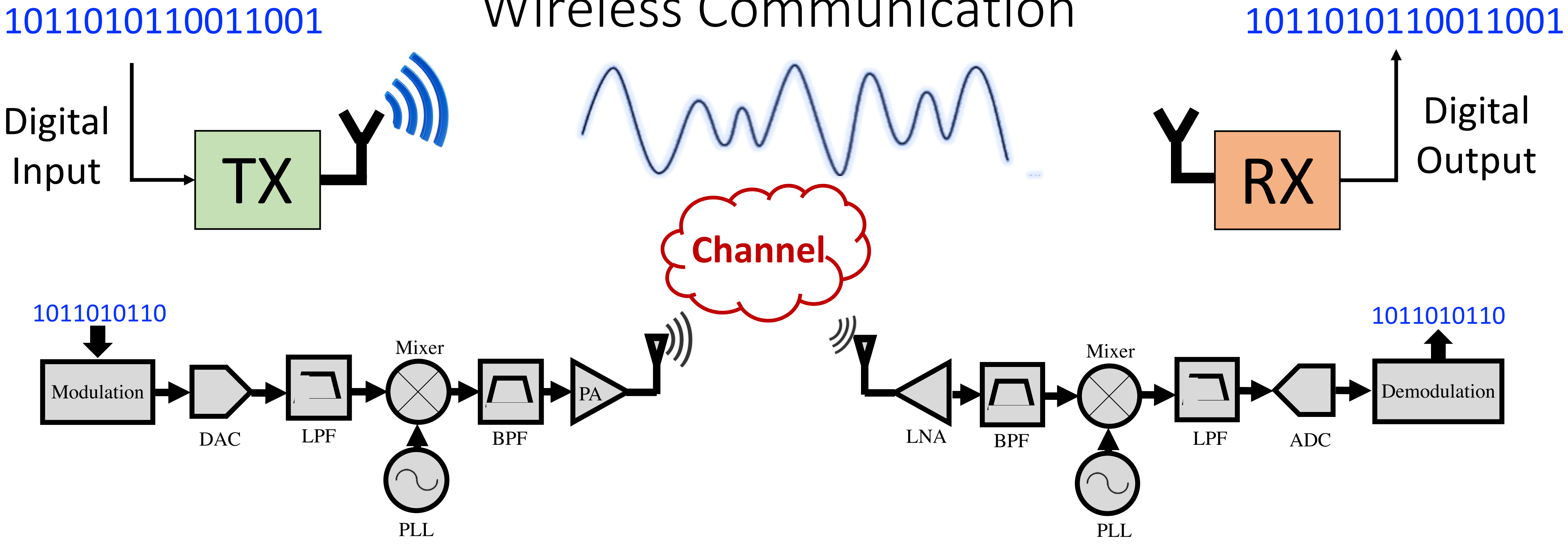
Haitham Hassanieh



EPFL

Laboratory of SEnsing
& Networking Systems

Wireless Communication

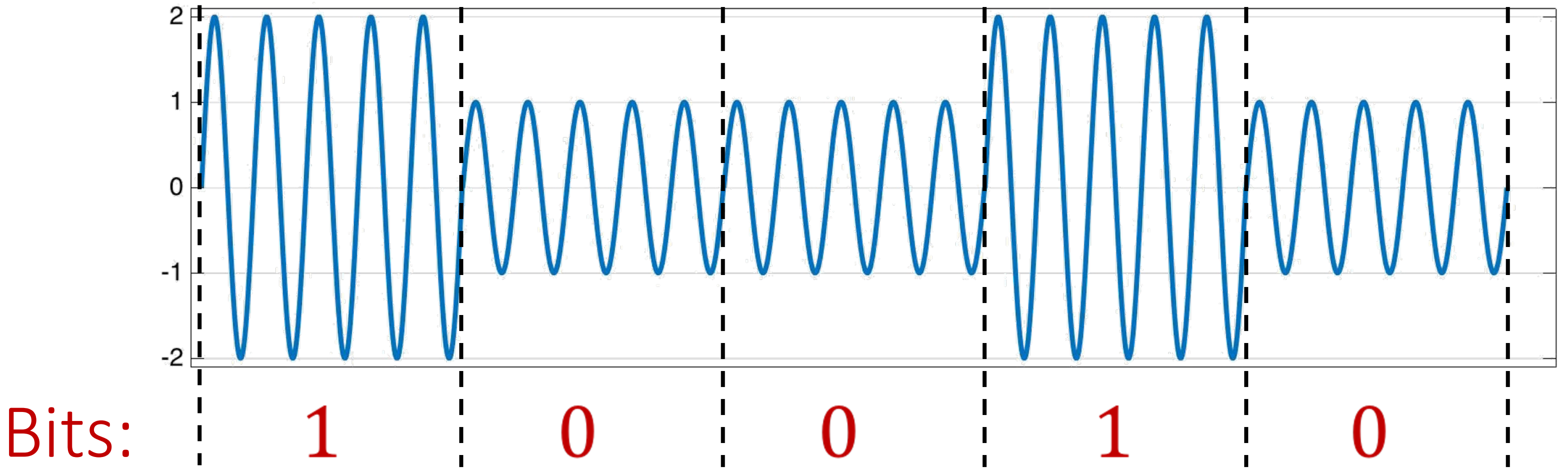


Modulation

Based on how the bits are encoded

ASK

Amplitude
Shift Keying

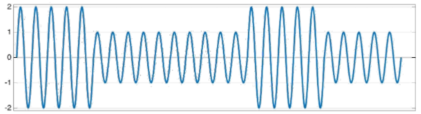


Modulation

Based on how the bits are encoded

ASK

Amplitude Shift Keying



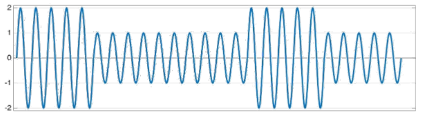
1 0 0 1 0

Modulation

Based on how the bits are encoded

ASK

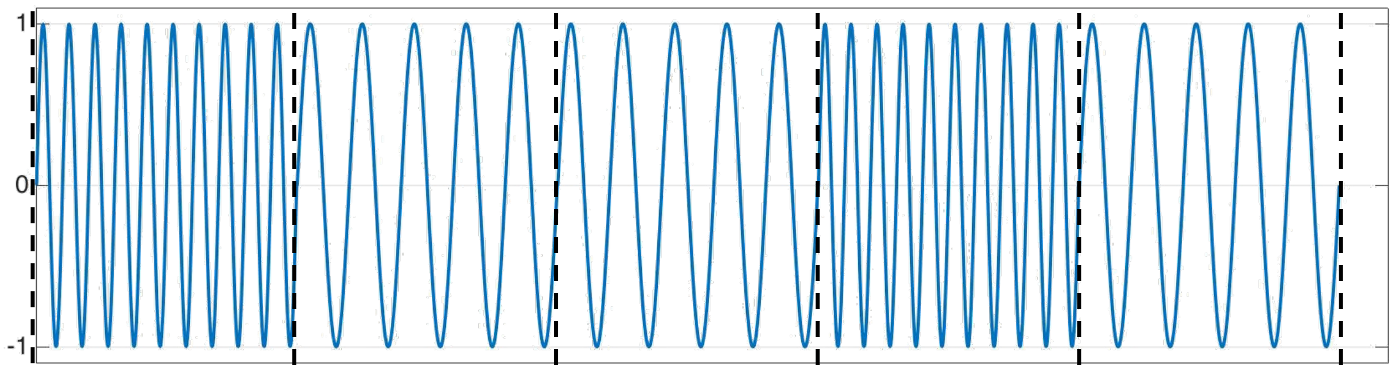
Amplitude
Shift Keying



1 0 0 1 0

FSK

Frequency
Shift Keying



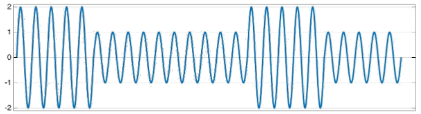
Bits: 1 0 0 1 0

Modulation

Based on how the bits are encoded

ASK

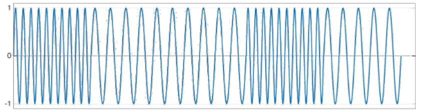
Amplitude Shift Keying



1 0 0 1 0

FSK

Frequency Shift Keying



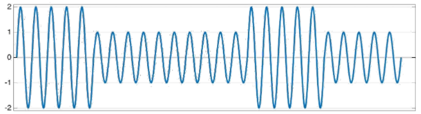
1 0 0 1 0

Modulation

Based on how the bits are encoded

ASK

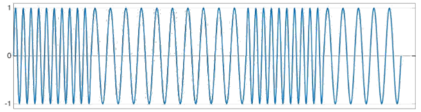
Amplitude
Shift Keying



1 0 0 1 0

FSK

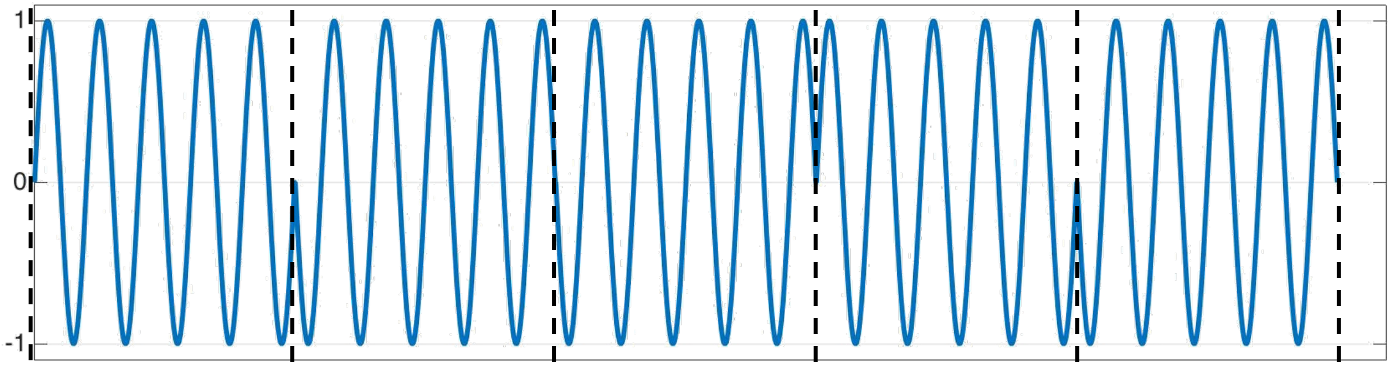
Frequency
Shift Keying



1 0 0 1 0

PSK

Phase
Shift Keying



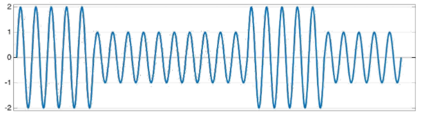
Bits: 1 0 0 1 0

Modulation

Based on how the bits are encoded

ASK

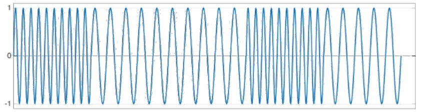
Amplitude
Shift Keying



1 0 0 1 0

FSK

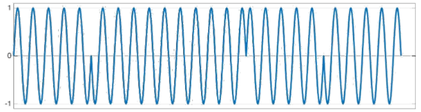
Frequency
Shift Keying



1 0 0 1 0

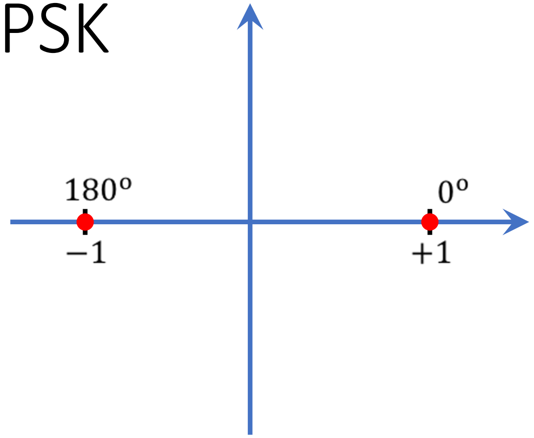
PSK

Phase
Shift Keying



1 0 0 1 0

BPSK

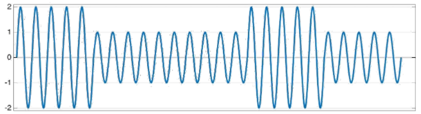


Modulation

Based on how the bits are encoded

ASK

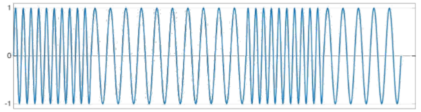
Amplitude
Shift Keying



1 0 0 1 0

FSK

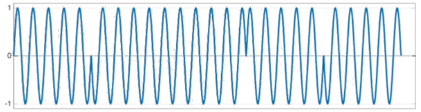
Frequency
Shift Keying



1 0 0 1 0

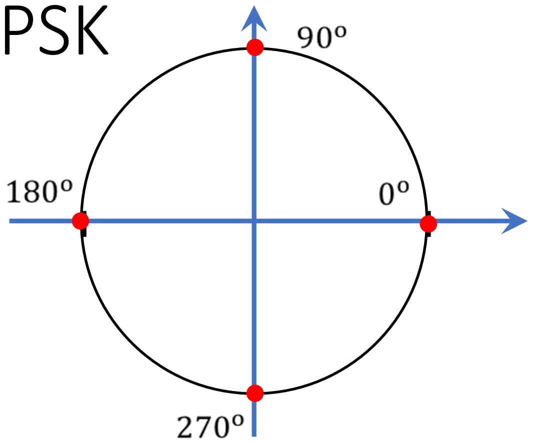
PSK

Phase
Shift Keying



1 0 0 1 0

4-PSK

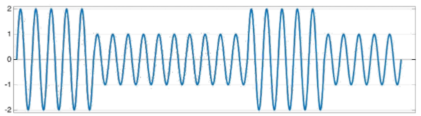


Modulation

Based on how the bits are encoded

ASK

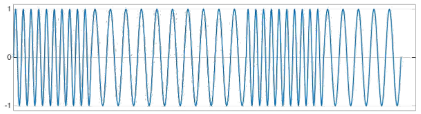
Amplitude Shift Keying



1 0 0 1 0

FSK

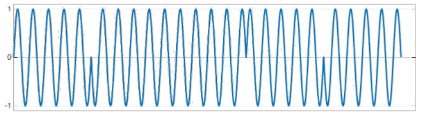
Frequency Shift Keying



1 0 0 1 0

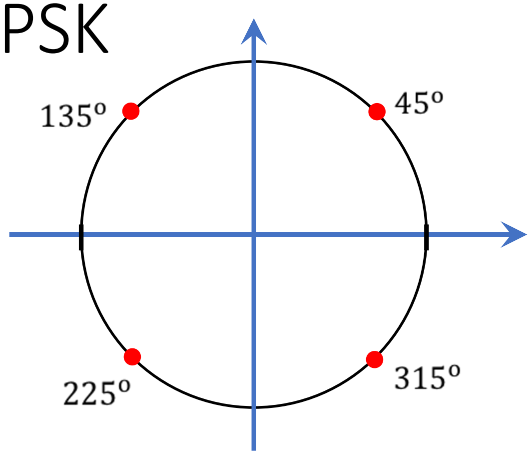
PSK

Phase Shift Keying



1 0 0 1 0

4-PSK

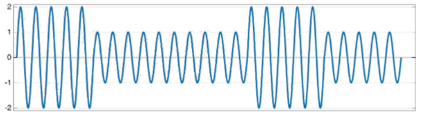


Modulation

Based on how the bits are encoded

ASK

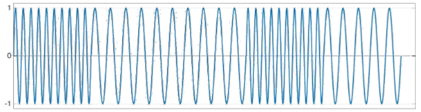
Amplitude
Shift Keying



1 0 0 1 0

FSK

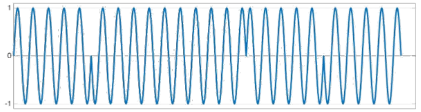
Frequency
Shift Keying



1 0 0 1 0

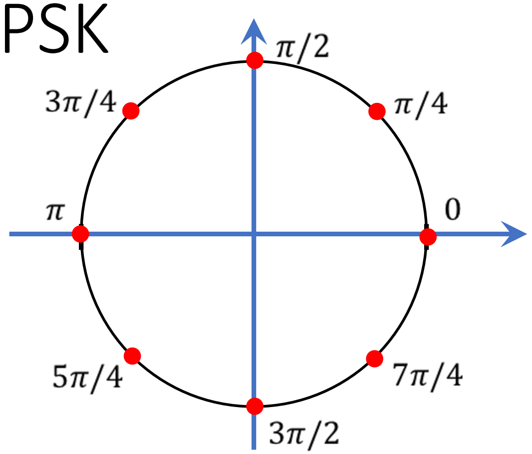
PSK

Phase
Shift Keying



1 0 0 1 0

8-PSK

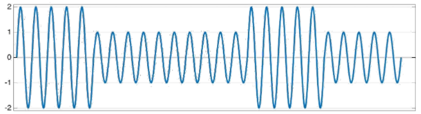


Modulation

Based on how the bits are encoded

ASK

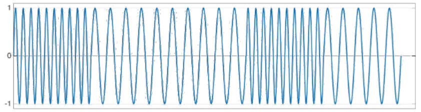
Amplitude Shift Keying



1 0 0 1 0

FSK

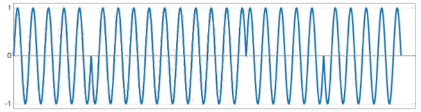
Frequency Shift Keying



1 0 0 1 0

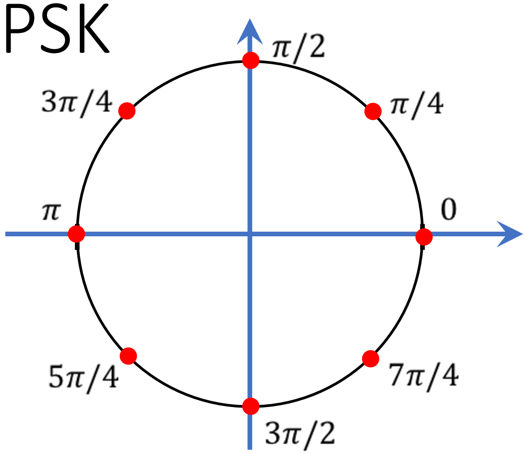
PSK

Phase Shift Keying



1 0 0 1 0

M-PSK



Phase is integer multiple of: $\frac{2\pi}{M}$

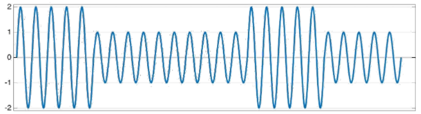
Bits per symbol: $\log_2 M$

Modulation

Based on how the bits are encoded

ASK

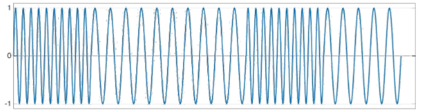
Amplitude Shift Keying



1 0 0 1 0

FSK

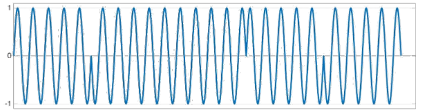
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying



1 0 0 1 0

QAM

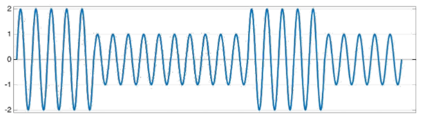
Phase & Amplitude Modulation (APSK)

Modulation

Based on how the bits are encoded

ASK

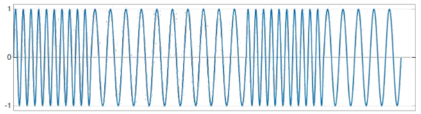
Amplitude Shift Keying



1 0 0 1 0

FSK

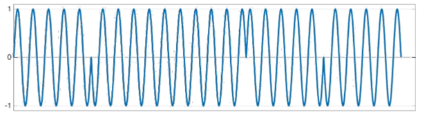
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying

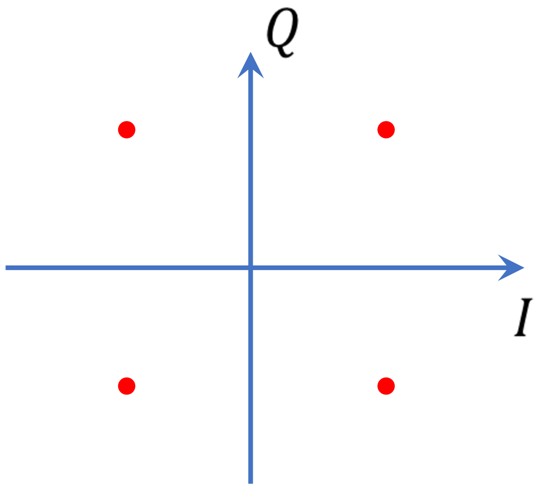


1 0 0 1 0

QAM

Phase & Amplitude Modulation (APSK)

4-QAM

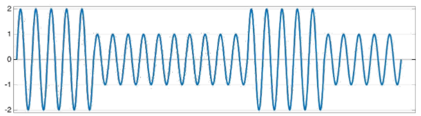


Modulation

Based on how the bits are encoded

ASK

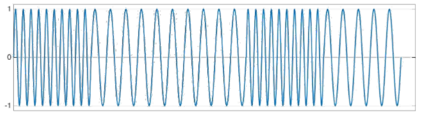
Amplitude Shift Keying



1 0 0 1 0

FSK

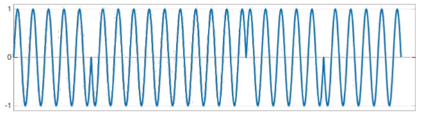
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying

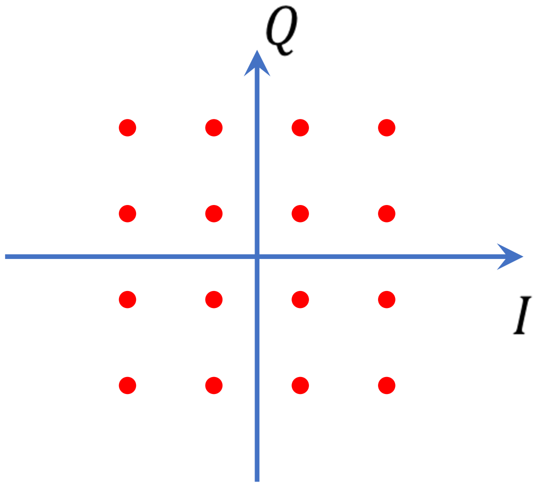


1 0 0 1 0

QAM

Phase & Amplitude Modulation (APSK)

16-QAM

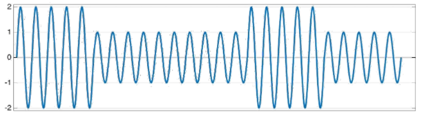


Modulation

Based on how the bits are encoded

ASK

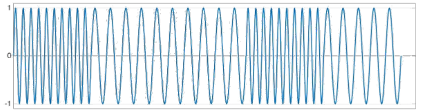
Amplitude Shift Keying



1 0 0 1 0

FSK

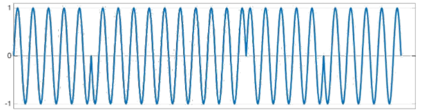
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying

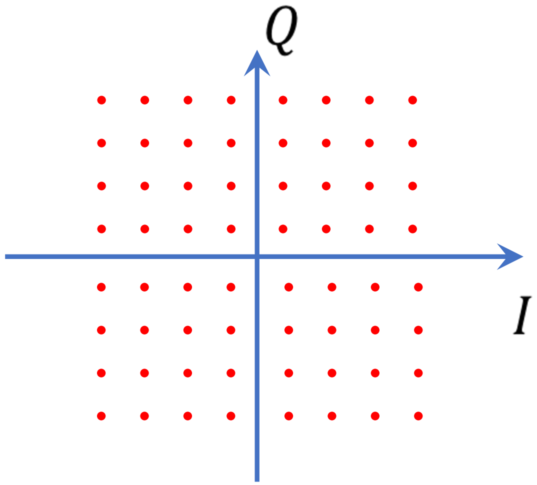


1 0 0 1 0

QAM

Phase & Amplitude Modulation (APSK)

64-QAM

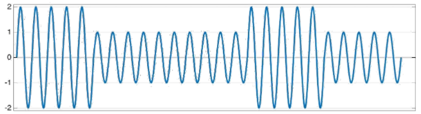


Modulation

Based on how the bits are encoded

ASK

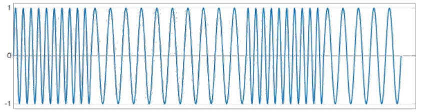
Amplitude Shift Keying



1 0 0 1 0

FSK

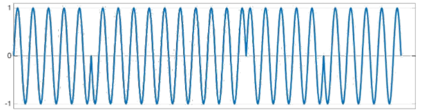
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying

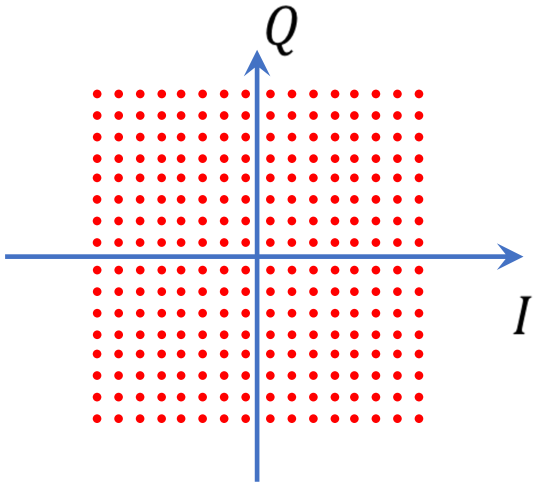


1 0 0 1 0

QAM

Phase & Amplitude Modulation (APSK)

256-QAM

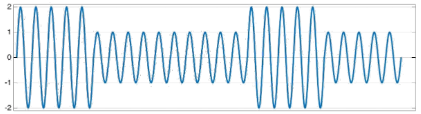


Modulation

Based on how the bits are encoded

ASK

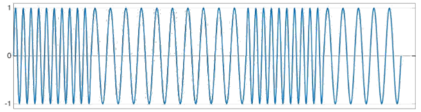
Amplitude Shift Keying



1 0 0 1 0

FSK

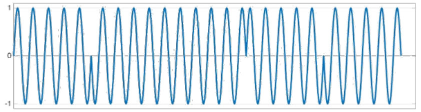
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying

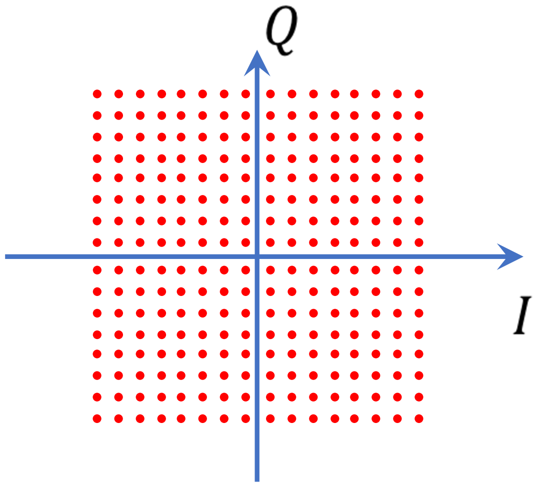


1 0 0 1 0

QAM

Phase & Amplitude Modulation (APSK)

M-QAM



IQ gird: $\sqrt{M}\times\sqrt{M}$

Bits per symbol: $\log_2 M$

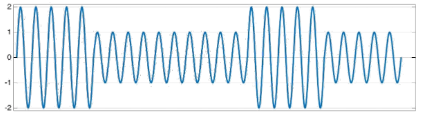
Modulation

Based on how the bits are encoded

ASK	FSK	PSK	QAM
Amplitude Shift Keying	Frequency Shift Keying	Phase Shift Keying	Phase & Amplitude Modulation (APSK)

ASK

Amplitude Shift Keying

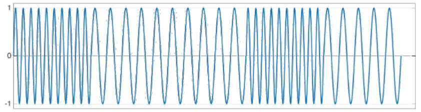


1 0 0 1 0

OOK, ASK

FSK

Frequency Shift Keying

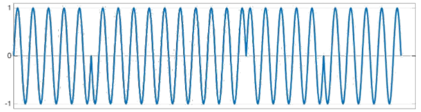


1 0 0 1 0

CFSK, MSK, GMSK

PSK

Phase Shift Keying



1 0 0 1 0

BPSK, QPSK, CPM

QAM

Phase & Amplitude Modulation (APSK)

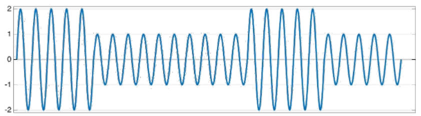
PAM, M-QAM

Modulation

Based on how the bits are encoded

ASK

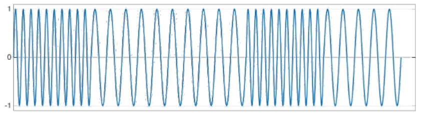
Amplitude Shift Keying



1 0 0 1 0

FSK

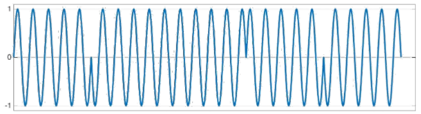
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying



1 0 0 1 0

QAM

Phase & Amplitude Modulation (APSK)

PPM

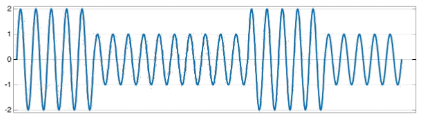
Pulse Position Modulation

Modulation

Based on how the bits are encoded

ASK

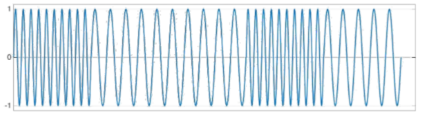
Amplitude Shift Keying



1 0 0 1 0

FSK

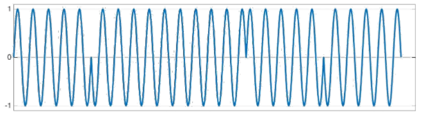
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying



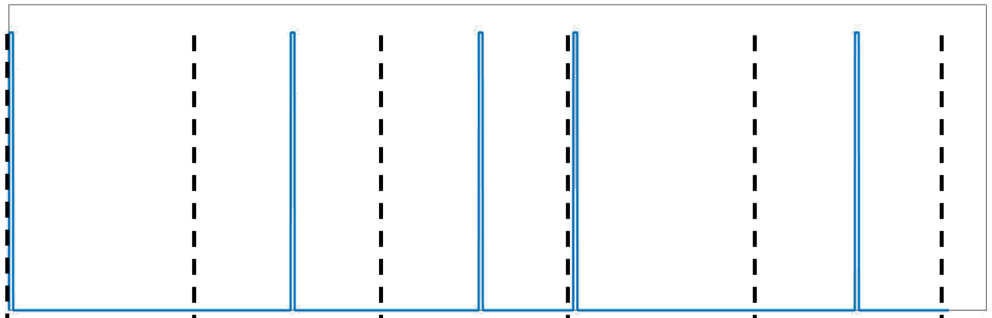
1 0 0 1 0

QAM

Phase & Amplitude Modulation (APSK)

PPM

Pulse Position Modulation



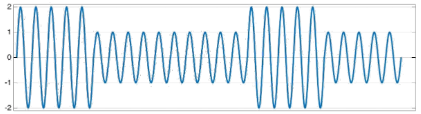
Bits: 1 0 0 1 0

Modulation

Based on how the bits are encoded

ASK

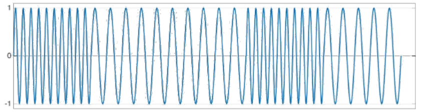
Amplitude Shift Keying



1 0 0 1 0

FSK

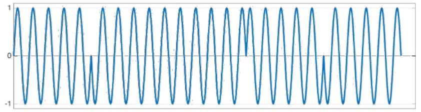
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying



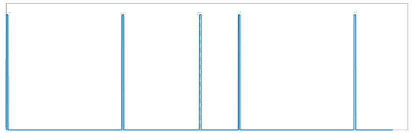
1 0 0 1 0

QAM

Phase & Amplitude Modulation (APSK)

PPM

Pulse Position Modulation



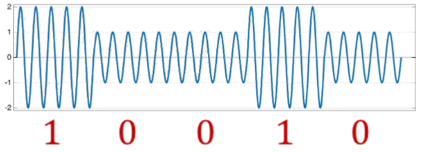
1 0 0 1 0

Modulation

Based on how the bits are encoded

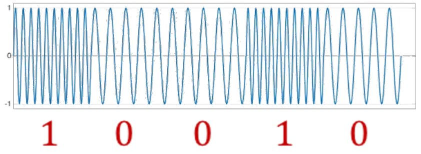
ASK

Amplitude Shift Keying



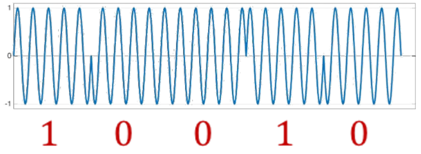
FSK

Frequency Shift Keying



PSK

Phase Shift Keying

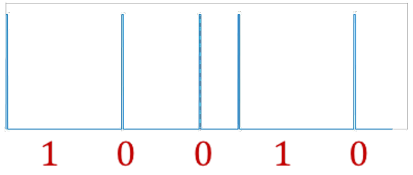


QAM

Phase & Amplitude Modulation (APSK)

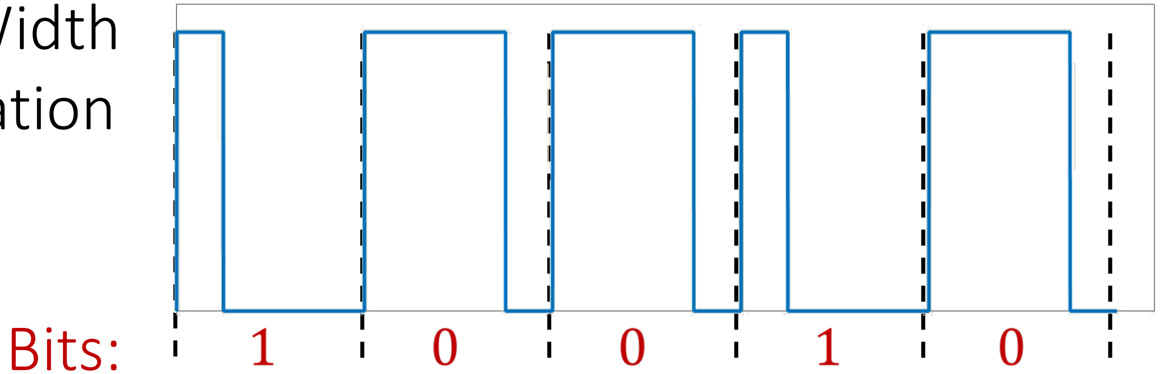
PPM

Pulse Position Modulation



PWM

Pulse Width Modulation

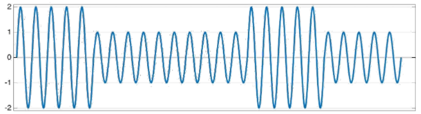


Modulation

Based on how the bits are encoded

ASK

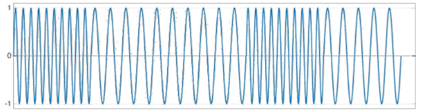
Amplitude Shift Keying



1 0 0 1 0

FSK

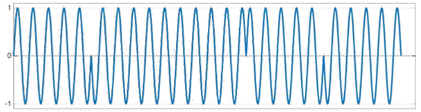
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying



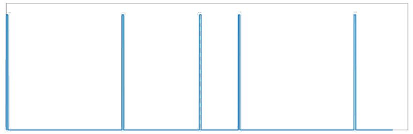
1 0 0 1 0

QAM

Phase & Amplitude Modulation (APSK)

PPM

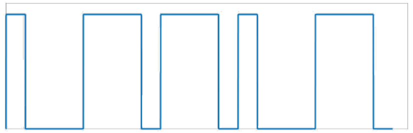
Pulse Position Modulation



1 0 0 1 0

PWM

Pulse Width Modulation



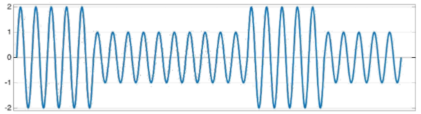
1 0 0 1 0

Modulation

Based on how the bits are encoded

ASK

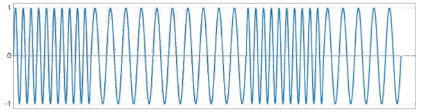
Amplitude Shift Keying



1 0 0 1 0

FSK

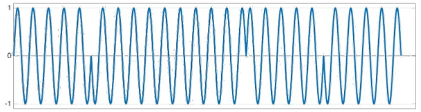
Frequency Shift Keying



1 0 0 1 0

PSK

Phase Shift Keying



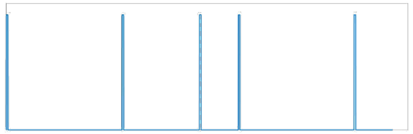
1 0 0 1 0

QAM

Phase & Amplitude Modulation (APSK)

PPM

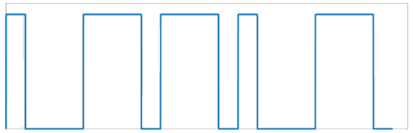
Pulse Position Modulation



1 0 0 1 0

PWM

Pulse Width Modulation



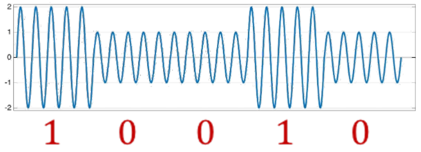
1 0 0 1 0

Modulation

Based on how the bits are encoded

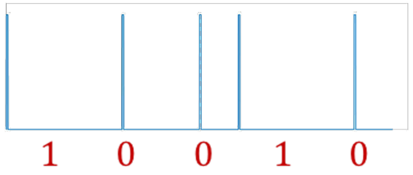
ASK

Amplitude Shift Keying



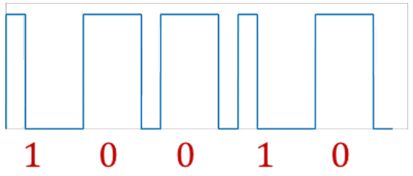
PPM

Pulse Position Modulation



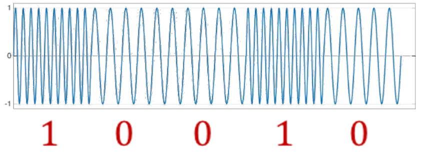
PWM

Pulse Width Modulation



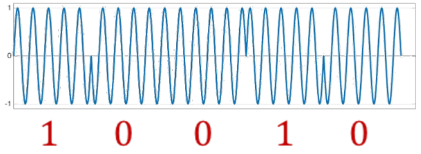
FSK

Frequency Shift Keying



PSK

Phase Shift Keying



DQPSK

Differential QPSK

QAM

Phase & Amplitude Modulation (APSK)

DQAM

Differential QAM

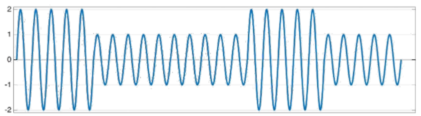
Encode bits as difference between consecutive symbols

Modulation

Based on how the bits are encoded

ASK

Amplitude Shift Keying

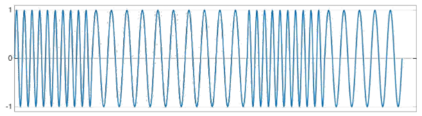


1 0 0 1 0

OOK, ASK

FSK

Frequency Shift Keying

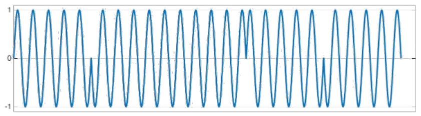


1 0 0 1 0

CFSK, MSK, GMSK

PSK

Phase Shift Keying



1 0 0 1 0

BPSK, QPSK, CPM

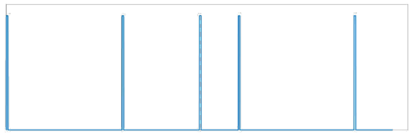
QAM

Phase & Amplitude Modulation (APSK)

PAM, M-QAM

PPM

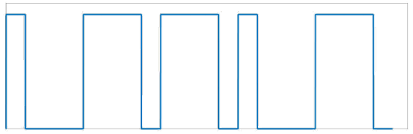
Pulse Position Modulation



1 0 0 1 0

PWM

Pulse Width Modulation



1 0 0 1 0

DQPSK

Differential QPSK

DBPSK, DQPSK,

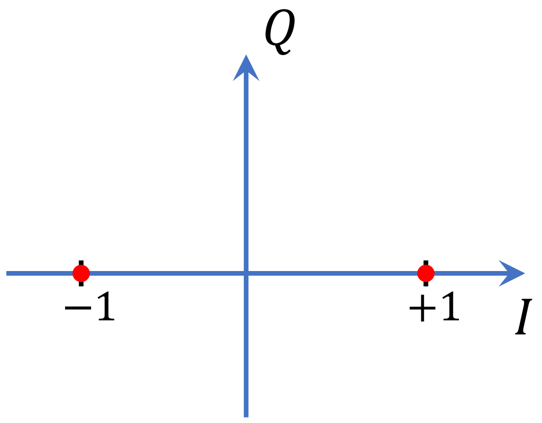
DQAM

Differential QAM

DQAM

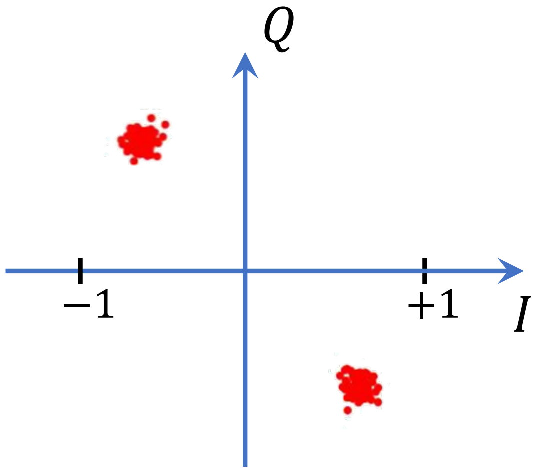
BPSK Modulation

$$0 \rightarrow -1$$
$$1 \rightarrow +1$$



$x(t)$

Transmitted Constellation

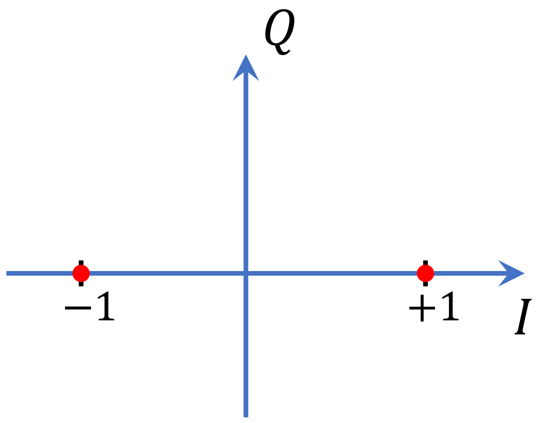


$y(t) = hx(t) + v(t)$

Received Constellation

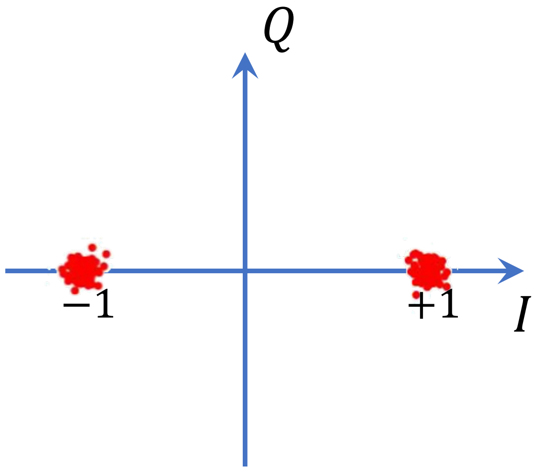
BPSK Modulation

$$0 \rightarrow -1$$
$$1 \rightarrow +1$$



$x(t)$

Transmitted Constellation

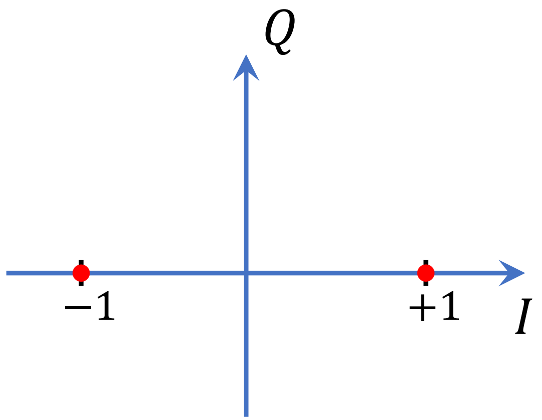


$y(t) = x(t) + v(t)/h$

Received Constellation

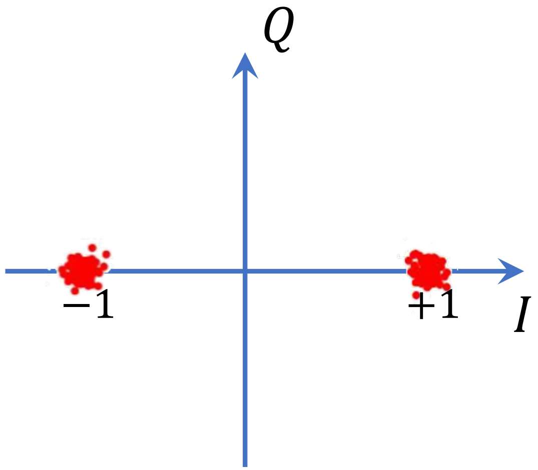
BPSK Modulation

$$0 \rightarrow -1$$
$$1 \rightarrow +1$$



$x(t)$

Transmitted Constellation



$y(t) = x(t) + v(t)/h$

Received Constellation

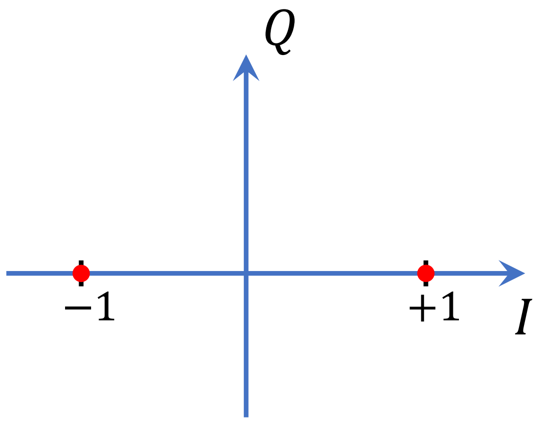
How well we can decode depends on SNR?

$$SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E[|hx(t)|^2]}{E[|v(t)|^2]} = \frac{|h|^2 E_s}{N'_0} = \frac{E_s}{N_0}$$

Lump attenuation in noise

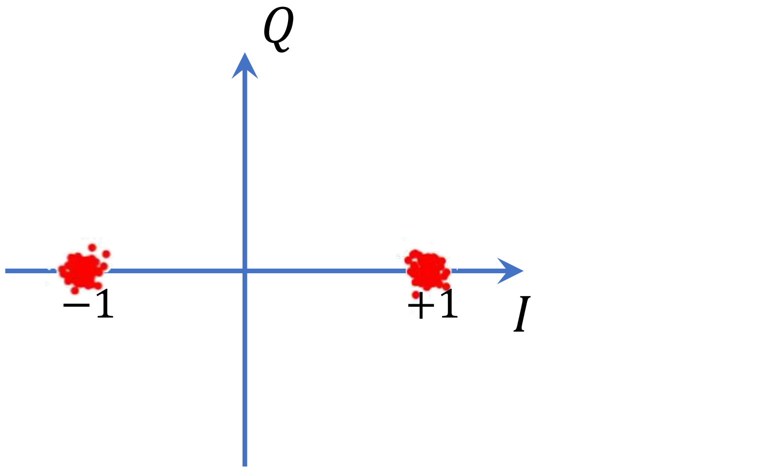
BPSK Modulation

$0 \rightarrow -1$
$1 \rightarrow +1$



$x(t)$

Transmitted Constellation



$y(t) = x(t) + v(t)$

Received Constellation

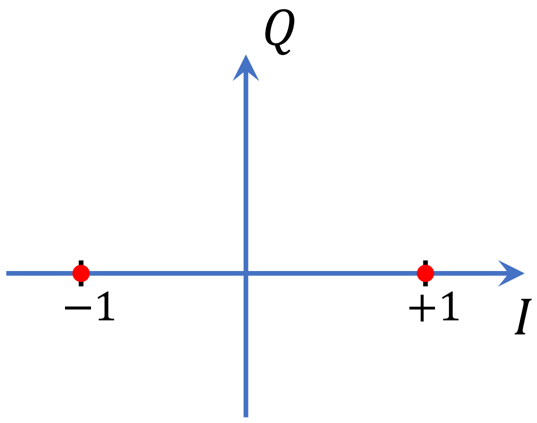
How well we can decode depends on SNR?

$$SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E_s}{N_0} = 25\ \mathrm{dB}$$

BPSK Modulation

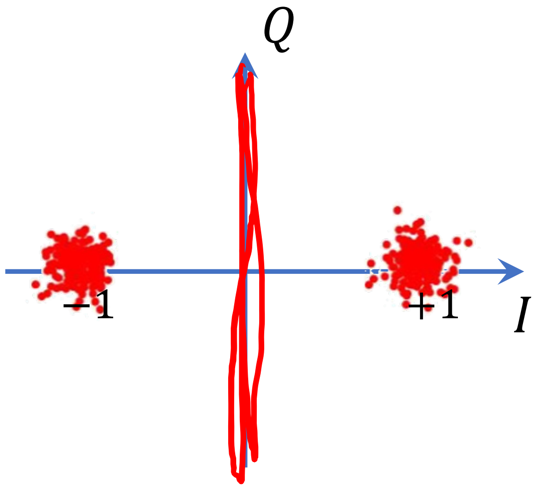
$0 \rightarrow -1$

$1 \rightarrow +1$



$x(t)$

Transmitted Constellation



$y(t) = x(t) + v(t)$ AWGN

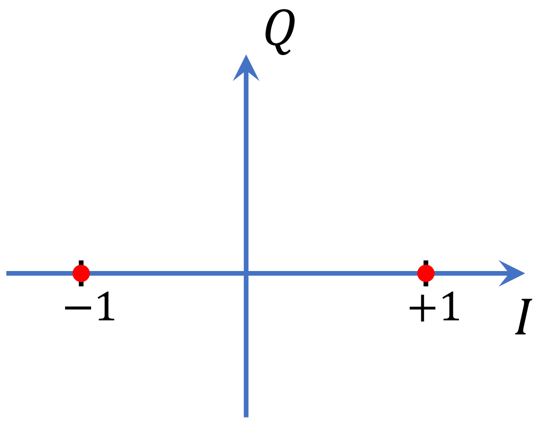
Received Constellation

How well we can decode depends on SNR?

$$SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E_s}{N_0} = 19\ \text{dB}$$

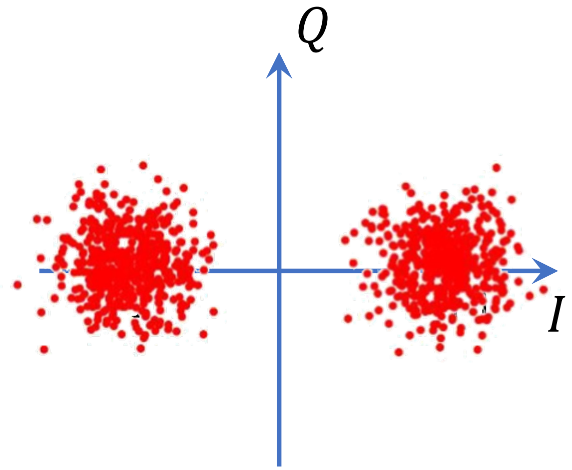
BPSK Modulation

$$0 \rightarrow -1$$
$$1 \rightarrow +1$$



$x(t)$

Transmitted Constellation



$y(t) = x(t) + v(t)$ AWGN

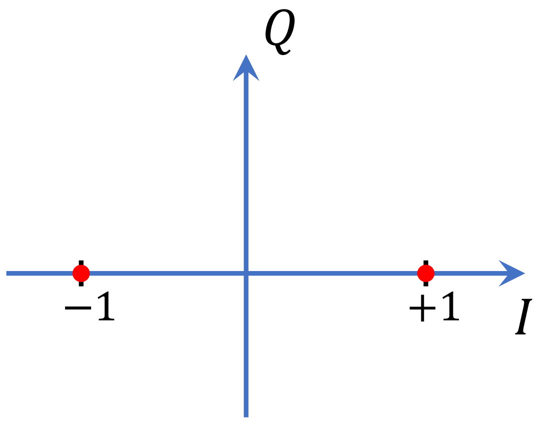
Received Constellation

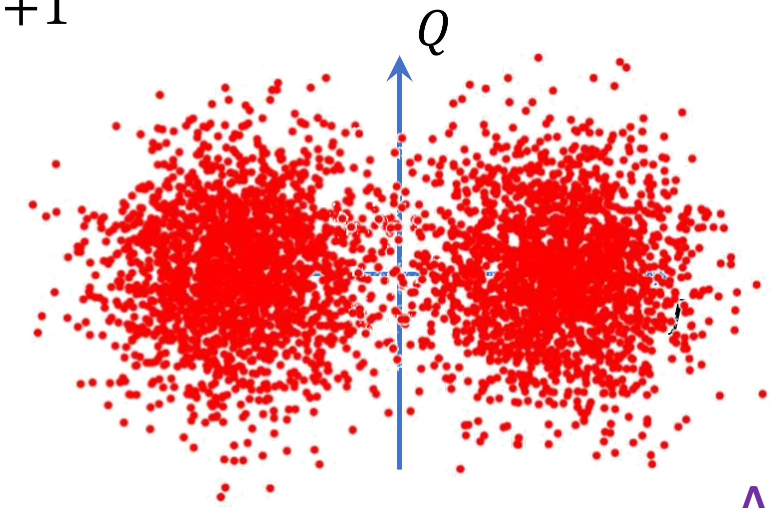
How well we can decode depends on SNR?

$$SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E_s}{N_0} = 13\ \text{dB}$$

BPSK Modulation

$$0 \rightarrow -1$$
$$1 \rightarrow +1$$





$x(t)$

Transmitted Constellation

$y(t) = x(t) + v(t)$ ← AWGN

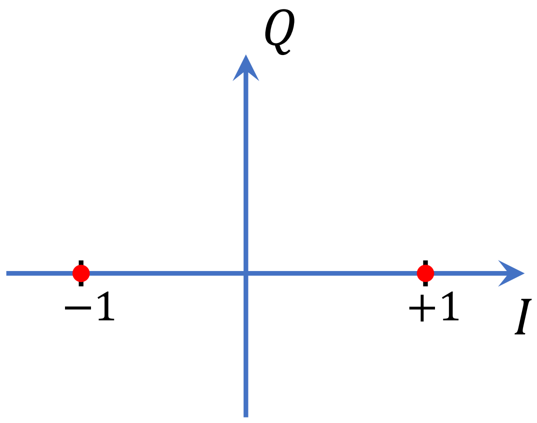
Received Constellation

How well we can decode depends on SNR?

$$SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E_s}{N_0} = 7\ \text{dB}$$

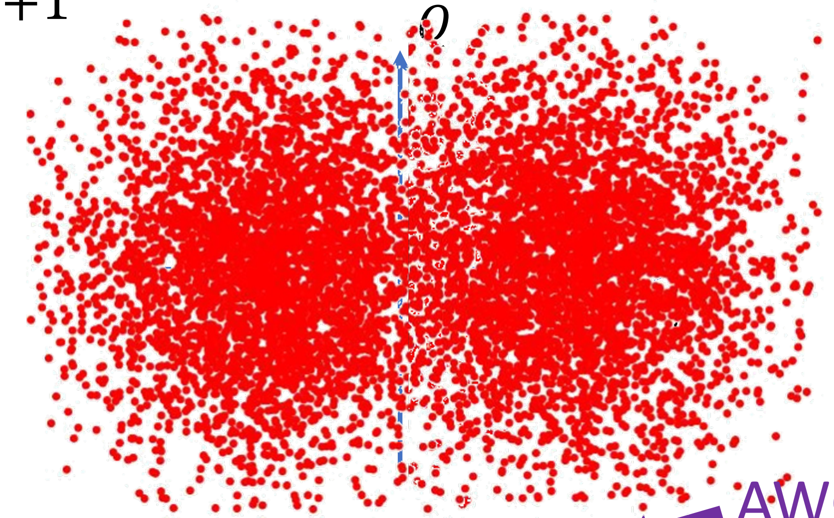
BPSK Modulation

$0 \rightarrow -1$
$1 \rightarrow +1$



$x(t)$

Transmitted Constellation



AWGN

$y(t) = x(t) + v(t)$

Received Constellation

How well we can decode depends on SNR?

$$SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E_s}{N_0} = 3.5\ \text{dB}$$

Hypothesis Testing

$$x = \pm 1 \longrightarrow y = x + v$$

$$P(x = +1|y) \underset{0}{\overset{1}{\gtrless}} P(x = -1|y)$$

Hypothesis Testing

$$x = \pm 1 \longrightarrow y = x + v$$

$$P(x = +1|y) \underset{0}{\overset{1}{\gtrless}} P(x = -1|y)$$

$$P(x|y) = ?$$

$A.\ \dfrac{P(x \cup y)}{P(y)}$ $\quad B.\ \dfrac{P(x \cap y)}{P(y)}$ $\quad C.\ \dfrac{P(x \cap y)}{P(x)}$ $\quad D.\ \dfrac{P(x)P(y|\, x)}{P(y)}$ $\quad E.\ \dfrac{P(y)P(y|\, x)}{P(x)}$

Hypothesis Testing

$$x = \pm 1 \longrightarrow y = x + v$$

$$P(x = +1|y) \underset{0}{\overset{1}{\gtrless}} P(x = -1|y)$$

$P(x|y) =?$

$A.\ \dfrac{P(x \cup y)}{P(y)}$ $B.\ \dfrac{P(x \cap y)}{P(y)}$ $C.\ \dfrac{P(x \cap y)}{P(x)}$ $D.\ \dfrac{P(x)P(y|\,x)}{P(y)}$ $E.\ \dfrac{P(y)P(y|\,x)}{P(x)}$

Hypothesis Testing

$$x = \pm 1 \longrightarrow y = x + v$$

$$P(x = +1|y) \underset{0}{\overset{1}{\gtrless}} P(x = -1|y)$$

$$\frac{\cancel{P(x = +1)}\times P(y|\, x = +1)}{\cancel{P(y)}} \underset{0}{\overset{1}{\gtrless}} \frac{\cancel{P(x = -1)}P(y|\, x = -1)}{\cancel{P(y)}}$$

$$\boxed{P(x = +1) = P(x = -1) = \frac{1}{2}}$$

$$P(y|x = +1) \underset{0}{\overset{1}{\gtrless}} P(y|x = -1)$$

Maximum Likelihood Decoder

$$x = \pm 1 \longrightarrow y = x + v$$

$y \sim CN(0,\sigma)$

$y + a \sim CN(a,\sigma)$

$$P(y|x=+1) \underset{0}{\overset{1}{\gtrless}} P(y|x=-1)$$

Guassian Noise: $v \sim \mathcal{CN}(0,\sigma) \longrightarrow y|x \sim \mathcal{CN}(x,\sigma)$

$$P(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|v|^2}{2\sigma^2}}$$

$$P(y|x=+1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|y-1|^2}{2\sigma^2}}$$

$$2\sigma^2 = N_0 \to \sigma = \sqrt{N_0/2}$$

$$P(y|x=-1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|y+1|^2}{2\sigma^2}}$$

Maximum Likelihood Decoder

$$x = \pm 1 \longrightarrow y = x + v$$

$$P(y|x = +1) \overset{1}{\underset{0}{\gtrless}} P(y|x = -1)$$

$$\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} e^{-\frac{|y-1|^2}{2\sigma^2}} \overset{1}{\underset{0}{\gtrless}} \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} e^{-\frac{|y+1|^2}{2\sigma^2}}$$

$$e^{-\frac{|y-1|^2}{2\sigma^2}} \overset{1}{\underset{0}{\gtrless}} e^{-\frac{|y+1|^2}{2\sigma^2}}$$

$$-\frac{|y-1|^2}{2\sigma^2} \overset{1}{\underset{0}{\gtrless}} -\frac{|y+1|^2}{2\sigma^2}$$

Maximum Likelihood Decoder

$$x = \pm 1 \longrightarrow y = x + v$$

$$P(y|x = +1) \underset{0}{\overset{1}{\gtrless}} P(y|x = -1)$$

$$-\frac{|y-1|^2}{2\sigma^2} \underset{0}{\overset{1}{\gtrless}} -\frac{|y+1|^2}{2\sigma^2}$$

$$-|y-1|^2 \underset{0}{\overset{1}{\gtrless}} -|y+1|^2$$

$$-(Re\{y\}-1)^2 - Im\{y\}^2 \underset{0}{\overset{1}{\gtrless}} -(Re\{y\}+1)^2 - Im\{y\}^2$$

Maximum Likelihood Decoder

$$x = \pm 1 \quad \longrightarrow \quad y = x + v$$

$$P(y|x = +1) \underset{0}{\overset{1}{\gtrless}} P(y|x = -1)$$

$$-(Re\{y\} - 1)^2 \underset{0}{\overset{1}{\gtrless}} -(Re\{y\} + 1)^2$$

$$-Re\{y\}^2 + 2Re\{y\} - 1 \underset{0}{\overset{1}{\gtrless}} -Re\{y\}^2 - 2Re\{y\} - 1$$

$$4Re\{y\} \underset{0}{\overset{1}{\gtrless}} 0$$

Maximum Likelihood Decoder

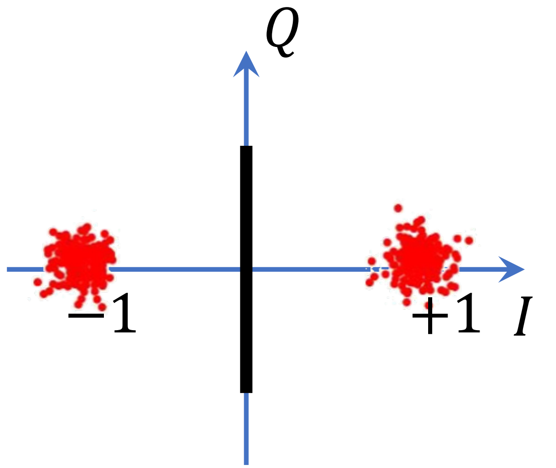
$x = \pm 1 \longrightarrow y = x + v$

$$P(y|x=+1) \underset{0}{\overset{1}{\gtrless}} P(y|x=-1)$$

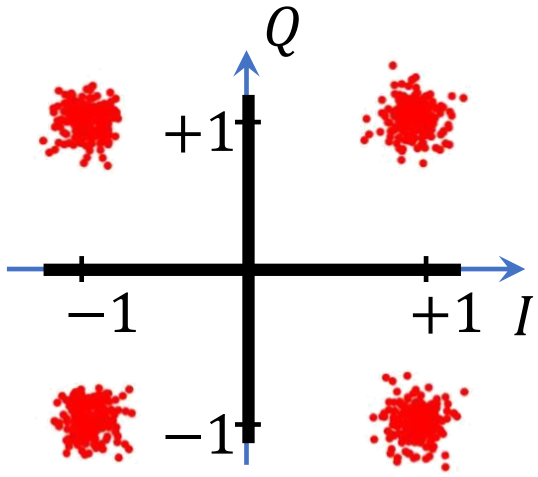
$$Re\{y\} \underset{0}{\overset{1}{\gtrless}} 0$$

$0 \rightarrow -1$

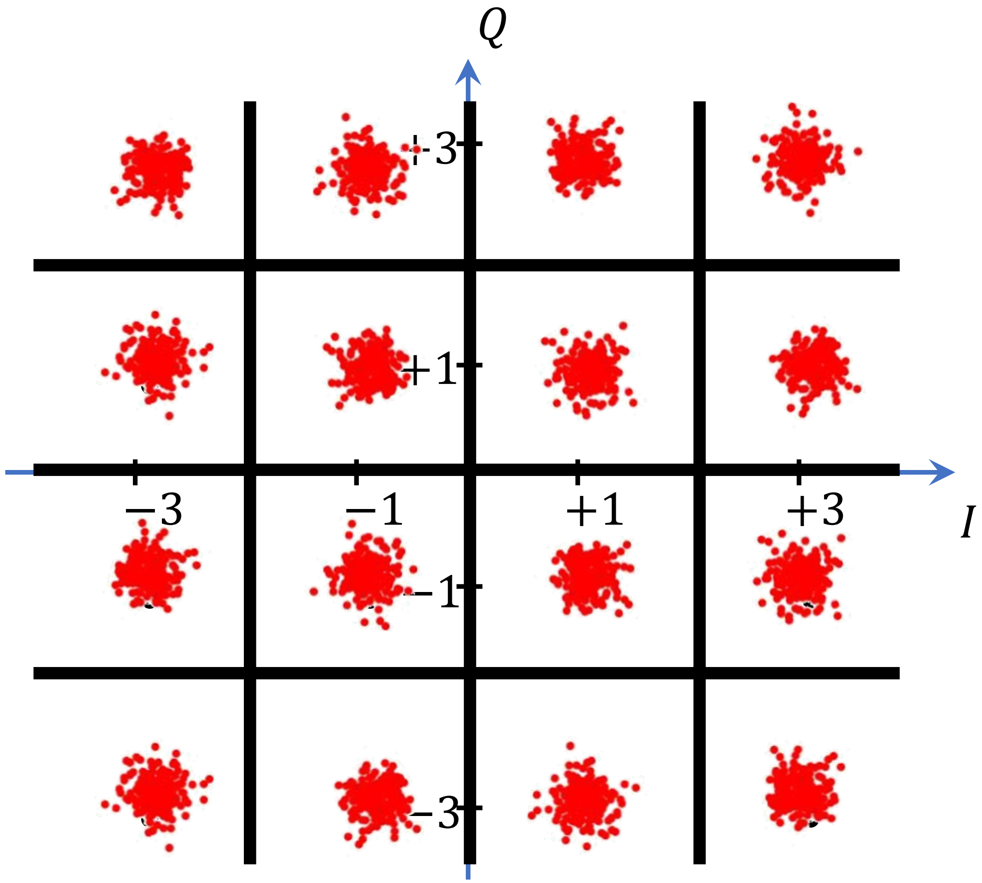
$1 \rightarrow +1$

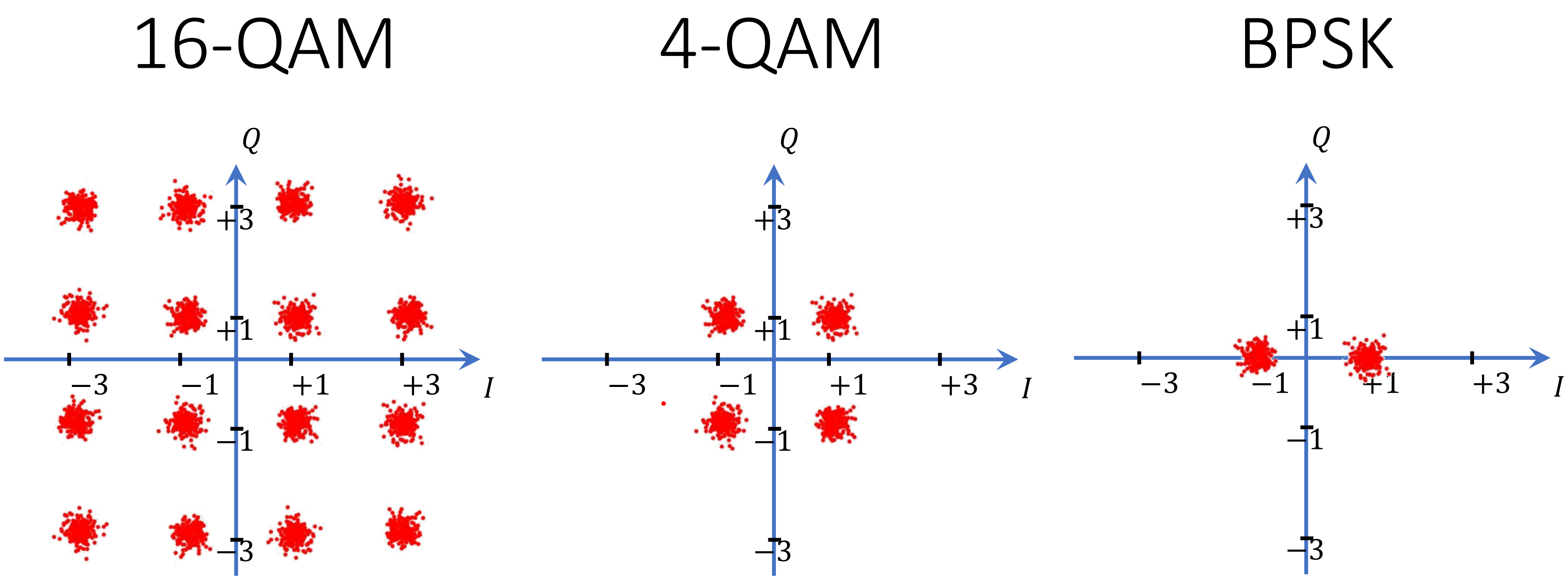


4-QAM



16-QAM

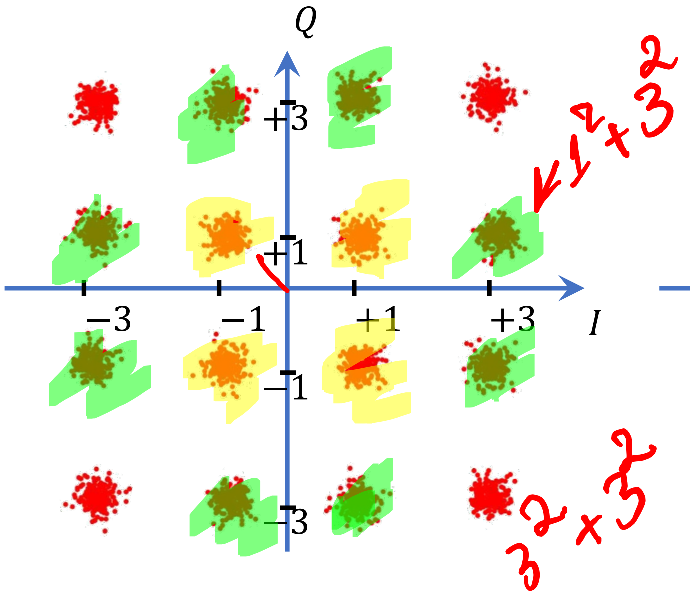




Is the SNR the same in these 3 constellations?

$$SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E[|x(t)|^2]}{E[|v(t)|^2]} = \frac{E_s}{N_0}$$

16-QAM

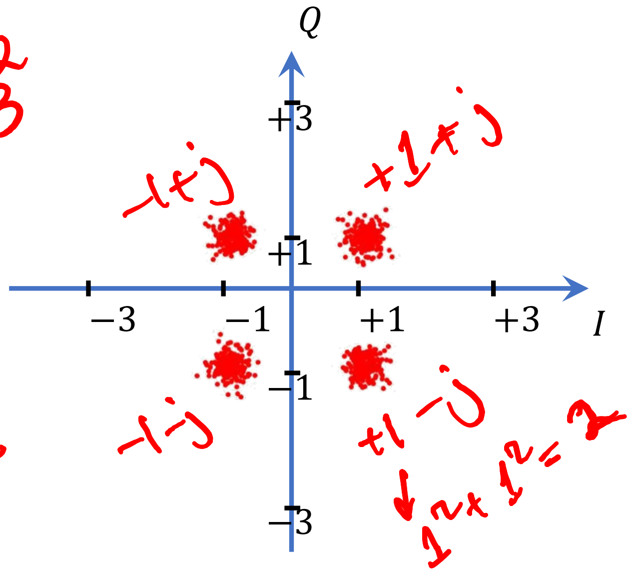


$$SNR = \frac{E_s(4\times 2 + 8\times 10 + 4\times 18)/16}{N_0}$$

$$= \frac{10E_s}{N_0}$$

$>$

4-QAM



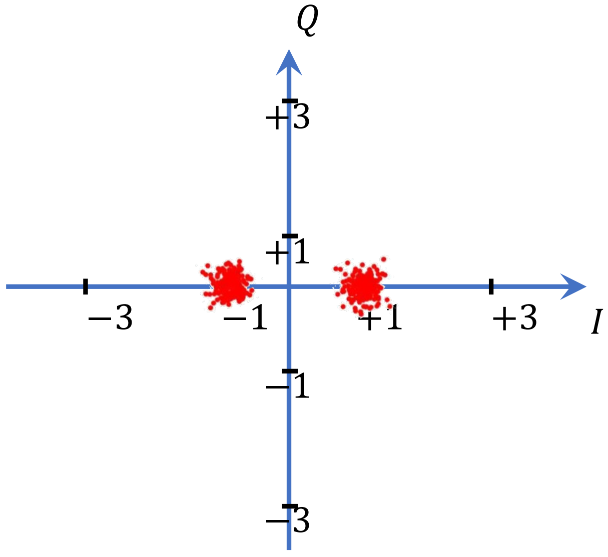
$$SNR = \frac{E[|x(t)|^2]}{N_0}$$

$$SNR = \frac{E_s(2+2+2+2)/4}{N_0}$$

$$= \frac{2E_s}{N_0}$$

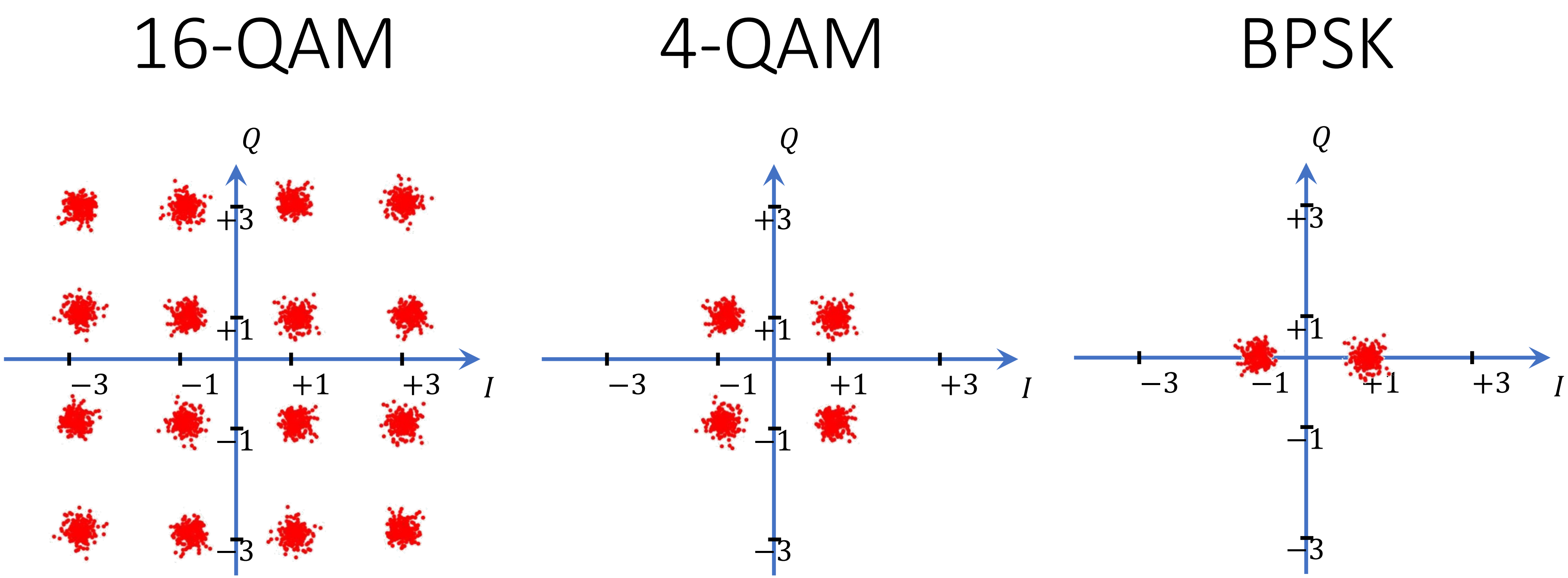
$>$

BPSK



$$SNR = \frac{(E_s+E_s)/2}{N_0}$$

$$= \frac{E_s}{N_0}$$



Transmit power is not the same!

$E_s(4\times2 + 8\times10$

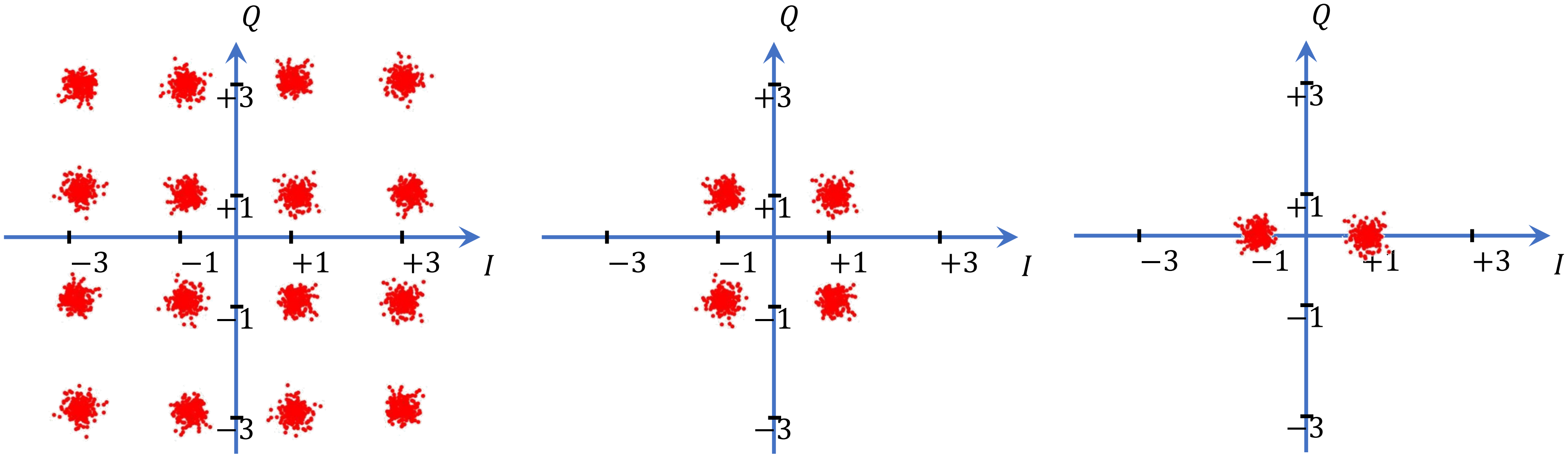
$+4$

$SNR =$ ... $s + E_s)/2$ / N_0

$= \frac{10E_s}{N_0}$ ... E_s / N_0

Must not increase transmit power to use higher order modulation!

16-QAM   4-QAM   BPSK



Normalize to maintain constant transmit power.

$$SNR = \frac{E_s(4\times 2 + 8\times 10 + 4\times 18)/16}{N_0} = \frac{10E_s}{N_0}$$

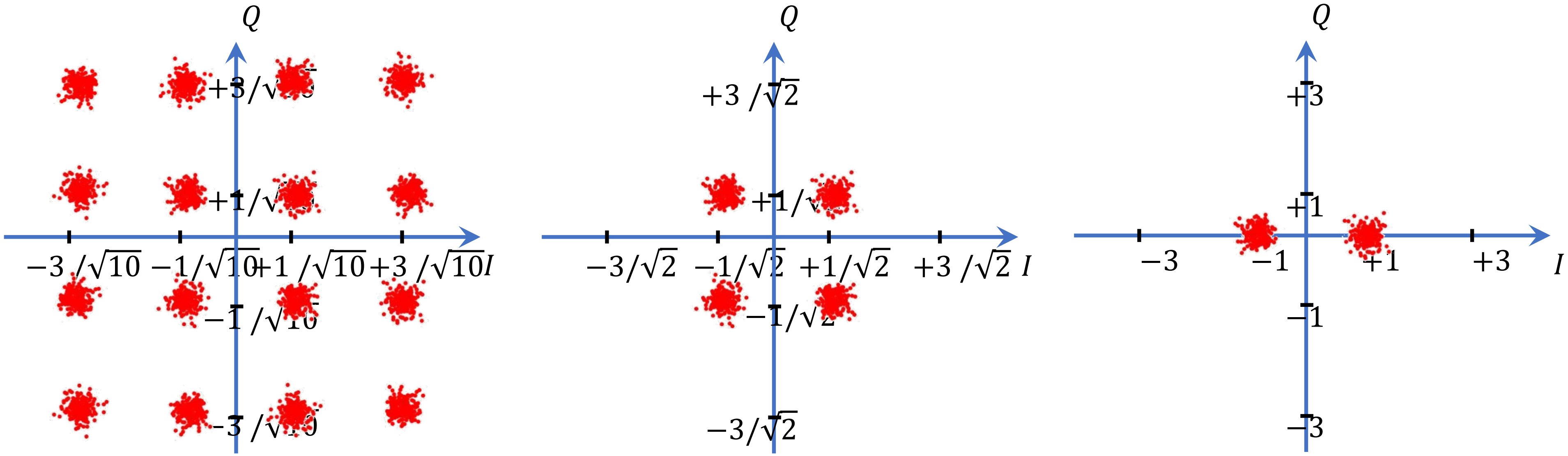
$>$

$$SNR = \frac{E_s(2+2+2+2)/4}{N_0} = \frac{2E_s}{N_0}$$

$>$

$$SNR = \frac{(E_s + E_s)/2}{N_0} = \frac{E_s}{N_0}$$

16-QAM 4-QAM BPSK



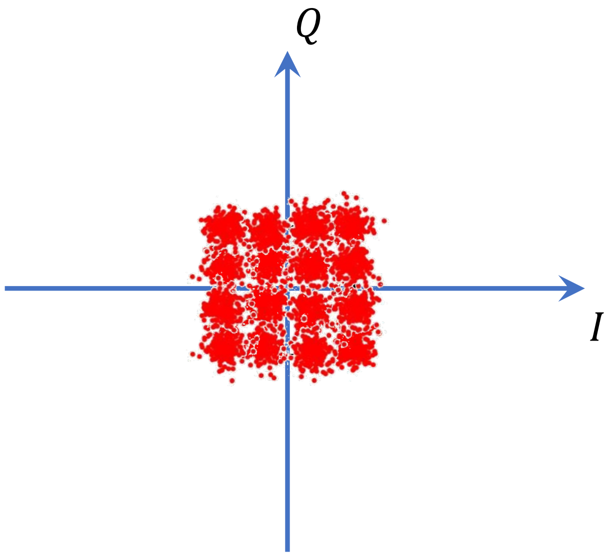
Normalize to maintain constant transmit power.

$$SNR = \frac{E_s(4\times 2/10 + 8\times 1 + 4\times 18/10)/16}{N_0} = \frac{E_s}{N_0}$$

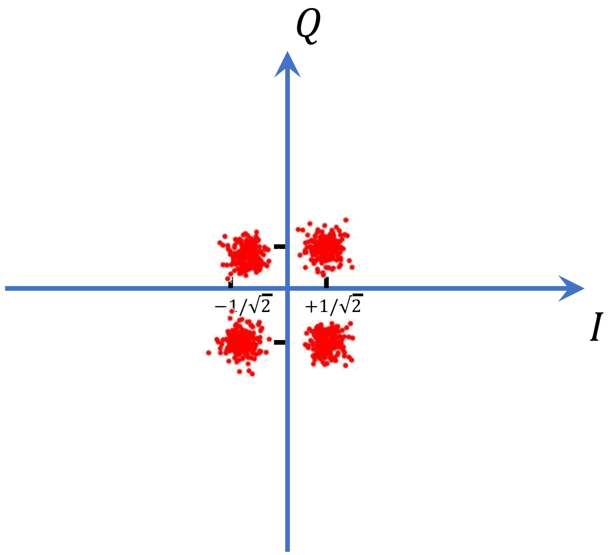
$$SNR = \frac{E_s(1+1+1+1)/4}{N_0} = \frac{E_s}{N_0}$$

$$SNR = \frac{(E_s + E_s)/2}{N_0} = \frac{E_s}{N_0}$$

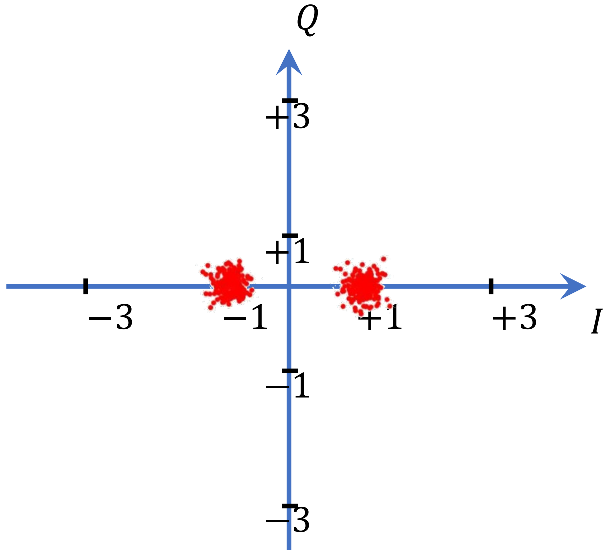
16-QAM



4-QAM



BPSK

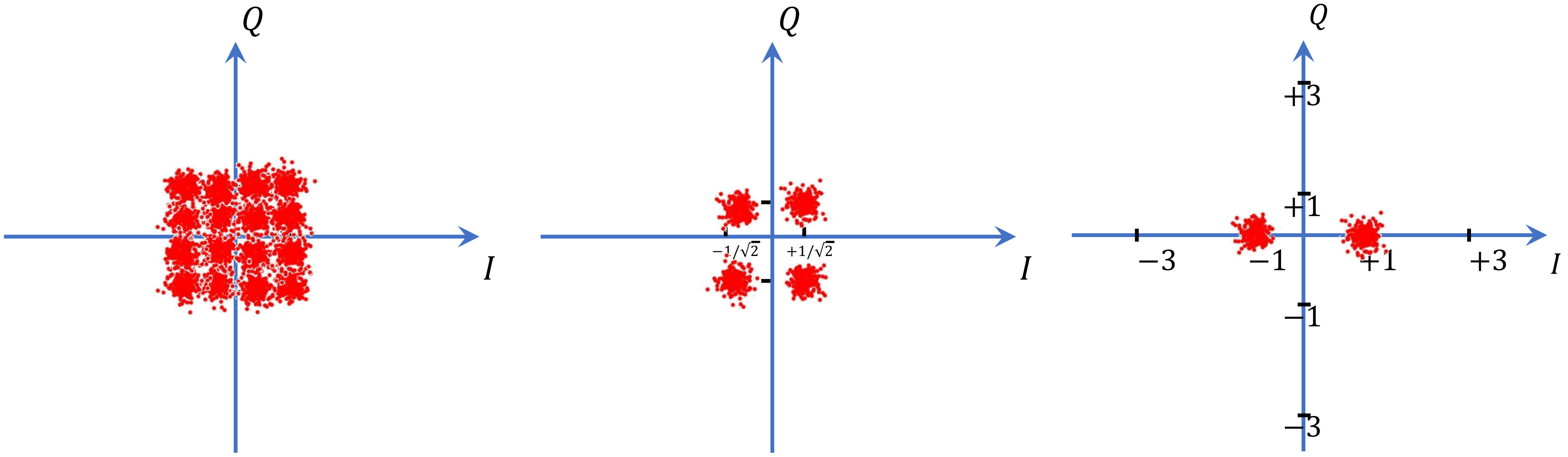


Normalize to maintain constant transmit power.

$$SNR = \frac{E_s}{N_0} \qquad SNR = \frac{E_s}{N_0} \qquad SNR = \frac{E_s}{N_0}$$

Need Higher SNR to Decode
Higher Order Modulation.

16-QAM 4-QAM BPSK



4 bits/symbol 2 bits/symbol 1 bit/symbol

Higher Order Modulation:

- Needs higher SNR to decode correctly.
- Achieves higher Bit Rate

Given an SNR, choose highest order modulation that guarantees minimal Bit Error Rate (BER)

Bit Error Rate (BER) of BPSK

Encoding:

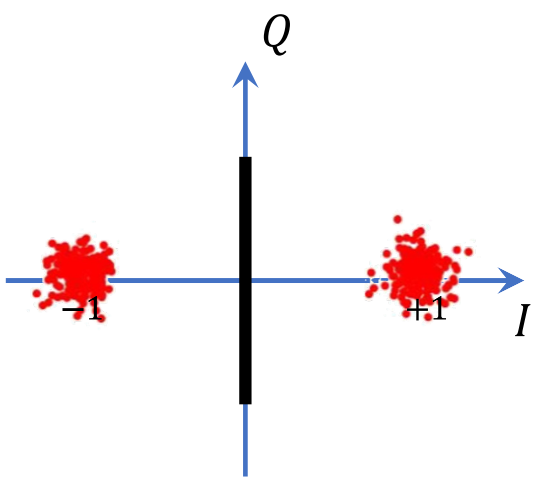
$$b = 0 \rightarrow x = -1$$
$$b = 1 \rightarrow x = +1$$

$$\longrightarrow \quad y = x + v$$

Decoding:

$$P(y|x = +1) \underset{0}{\overset{1}{\gtrless}} P(y|x = -1)$$

$$Re\{y\} \underset{0}{\overset{1}{\gtrless}} 0$$



Bit Error Rate (BER) of BPSK

Encoding:

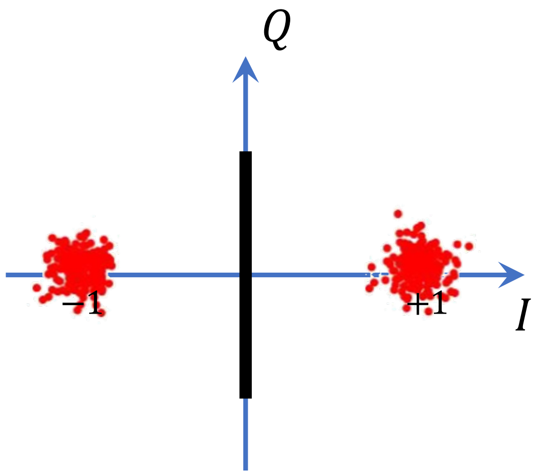
$$b = 0 \rightarrow x = -\sqrt{E_s}$$
$$b = 1 \rightarrow x = +\sqrt{E_s}$$

$$\longrightarrow \quad y = x + v$$

Decoding:

$$P\left(y|x = +\sqrt{E_s}\right) \overset{1}{\underset{0}{\gtrless}} P\left(y|x = -\sqrt{E_s}\right)$$

$$Re\{y\} \overset{1}{\underset{0}{\gtrless}} 0$$



$$BER = P\left(\hat{b} = 0 \cap b = 1\right) + P\left(\hat{b} = 1 \cap b = 0\right)$$

$$= P(b = 1)P\left(\hat{b} = 0|b = 1\right) + P(b = 0)P\left(\hat{b} = 1|b = 0\right)$$

$$= \frac{1}{2}P\left(\hat{b} = 0|b = 1\right) + \frac{1}{2}P\left(\hat{b} = 1|b = 0\right)$$

$$= \frac{1}{2}P(y < 0|b = 1) + \frac{1}{2}P(y > 0|b = 0)$$

$$= \frac{1}{2}P\left(y < 0|x = +\sqrt{E_s}\right) + \frac{1}{2}P\left(y > 0|x = -\sqrt{E_s}\right)$$

Bit Error Rate (BER) of BPSK

Encoding:

$$b = 0 \rightarrow x = -\sqrt{E_s}$$
$$b = 1 \rightarrow x = +\sqrt{E_s}$$

$$\longrightarrow \quad y = x + v$$

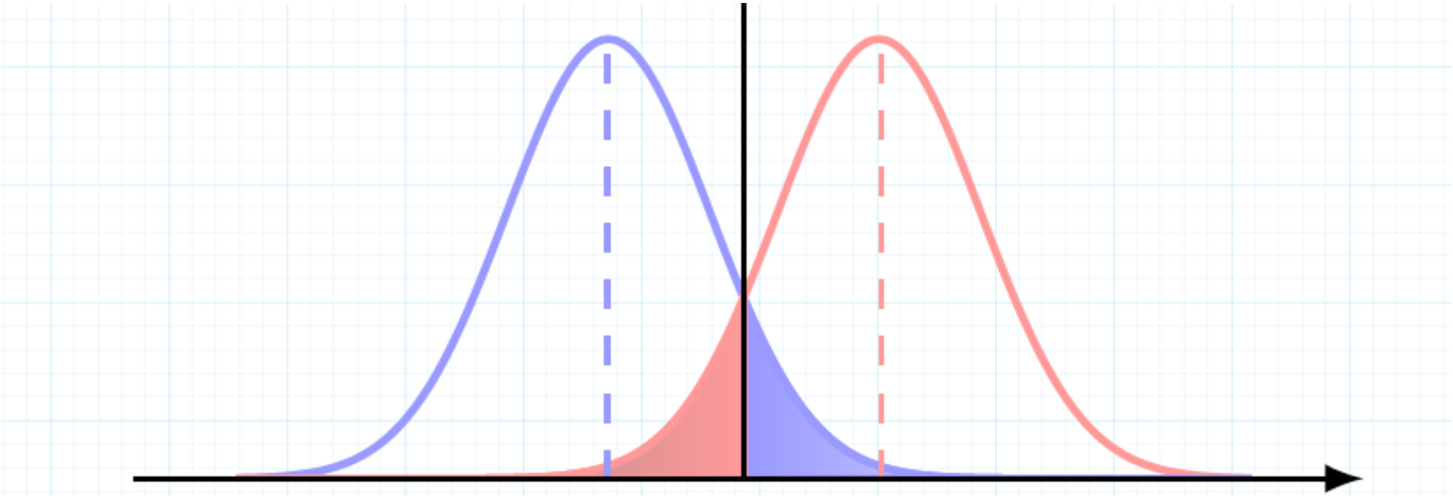
Decoding:

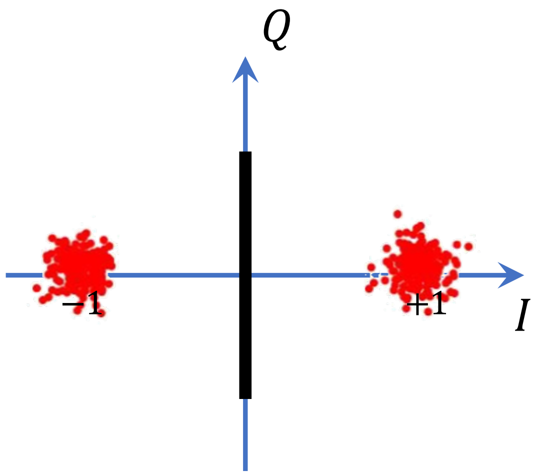
$$P\left(y|x = +\sqrt{E_s}\right) \underset{0}{\overset{1}{\gtrless}} P\left(y|x = -\sqrt{E_s}\right)$$

$$Re\{y\} \underset{0}{\overset{1}{\gtrless}} 0$$

$$BER = \frac{1}{2}P\left(y < 0|x = +\sqrt{E_s}\right) + \frac{1}{2}P\left(y > 0|x = -\sqrt{E_s}\right)$$

$$= \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(y-\sqrt{E_s}\right)^2}{2\sigma^2}} dy + \frac{1}{2}\int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(y+\sqrt{E_s}\right)^2}{2\sigma^2}} dy$$





Bit Error Rate (BER) of BPSK

Encoding:

$$b = 0 \rightarrow x = -\sqrt{E_s}$$
$$b = 1 \rightarrow x = +\sqrt{E_s}$$

$$\longrightarrow \quad y = x + v$$

Decoding:

$$P\left(y|x = +\sqrt{E_s}\right) \underset{0}{\overset{1}{\gtrless}} P\left(y|x = -\sqrt{E_s}\right)$$

$$Re\{y\} \underset{0}{\overset{1}{\gtrless}} 0$$

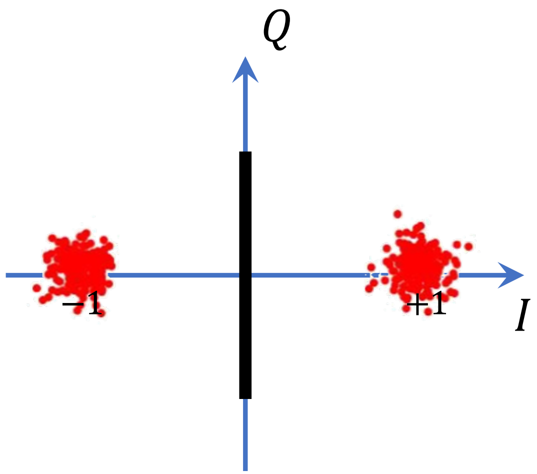
$$BER = \frac{1}{2}P\left(y < 0|x = +\sqrt{E_s}\right) + \frac{1}{2}P\left(y > 0|x = -\sqrt{E_s}\right)$$

$$= \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(y-\sqrt{E_s}\right)^2}{2\sigma^2}} dy + \frac{1}{2}\int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(y+\sqrt{E_s}\right)^2}{2\sigma^2}} dy$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(y+\sqrt{E_s}\right)^2}{2\sigma^2}} dy \qquad u = \frac{y + \sqrt{E_s}}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\sqrt{E_s}}{\sigma}}^{\infty} e^{-\frac{u^2}{2}} du = \frac{1}{2}\operatorname{erfc}\left(\frac{\sqrt{E_s}}{\sigma\sqrt{2}}\right) = Q\left(\sqrt{E_s}/\sigma\right) \boxed{= Q\left(\sqrt{2E_s/N_0}\right) = Q\left(\sqrt{2SNR}\right)}$$

Error Function Q Function



Bit Error Rate (BER) of BPSK

Encoding:

$$b = 0 \rightarrow x = -\sqrt{E_s}$$
$$b = 1 \rightarrow x = +\sqrt{E_s}$$

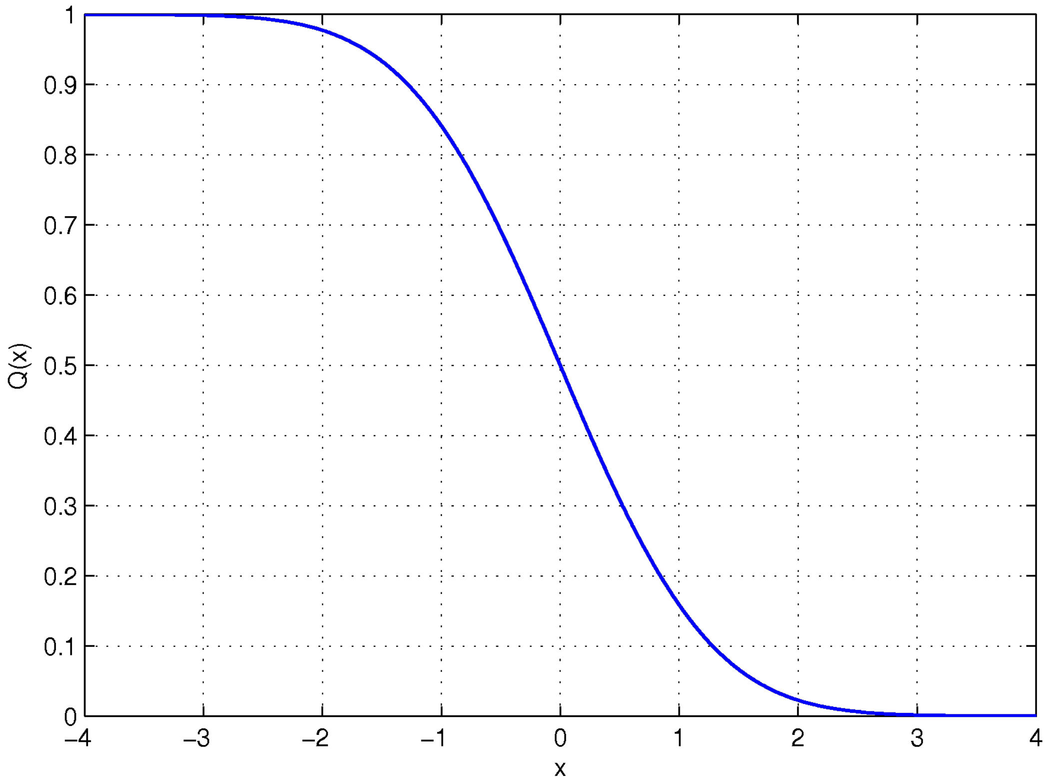
$$\longrightarrow \quad y = x + v$$

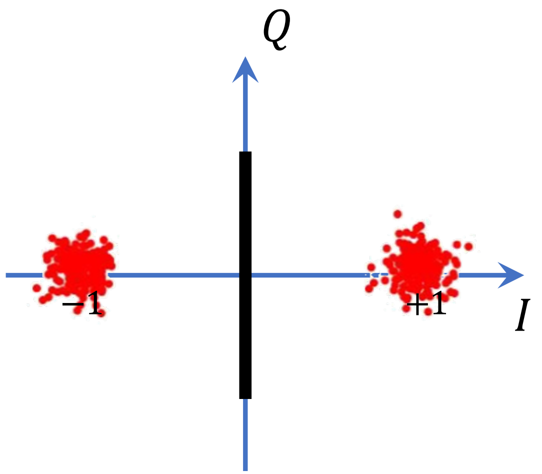
Decoding:

$$P\left(y|x = +\sqrt{E_s}\right) \underset{0}{\overset{1}{\gtrless}} P\left(y|x = -\sqrt{E_s}\right)$$

$$Re\{y\} \underset{0}{\overset{1}{\gtrless}} 0$$

$$BER \quad = Q\left(\sqrt{2SNR}\right) = Q\left(\sqrt{2E_s/N_0}\right)$$





Bit Error Rate (BER) of BPSK

Encoding:

$$b = 0 \rightarrow x = -\sqrt{E_s}$$
$$b = 1 \rightarrow x = +\sqrt{E_s}$$

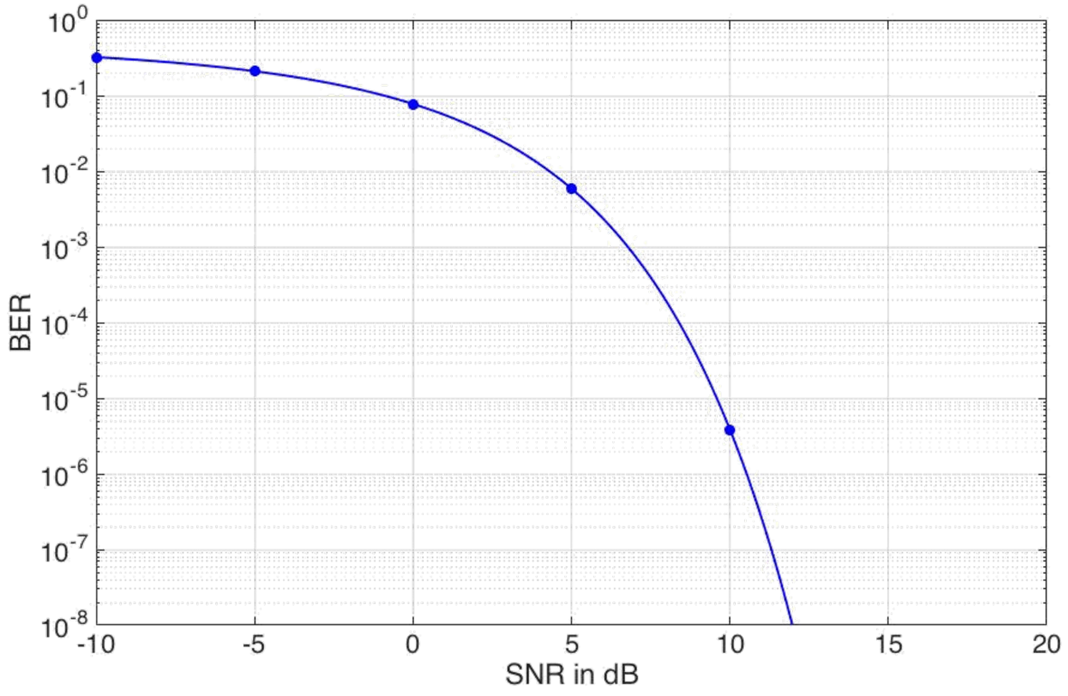
$$\longrightarrow \quad y = x + v$$

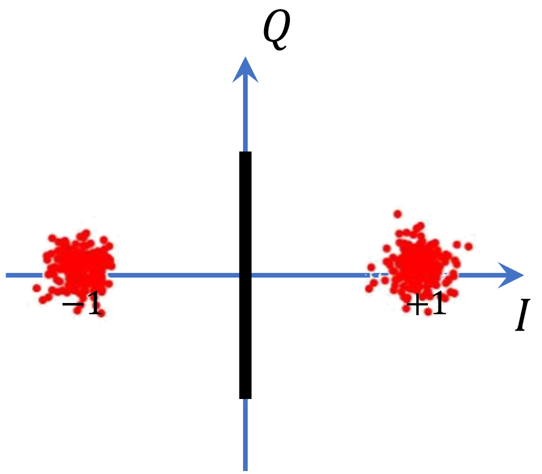
Decoding:

$$P\left(y|x = +\sqrt{E_s}\right) \underset{0}{\overset{1}{\gtrless}} P\left(y|x = -\sqrt{E_s}\right)$$

$$Re\{y\} \underset{0}{\overset{1}{\gtrless}} 0$$

$$BER = Q\left(\sqrt{2SNR}\right) = Q\left(\sqrt{2E_s/N_0}\right)$$





Bit Error Rate (BER)

- Number of Constellation Points: M
- Bits per symbol: $\log_2 M$
- Signal-to-Noise Ratio: SNR
- SNR per symbol: $\frac{E_s}{N_0}$
- SNR per bit: $\frac{E_b}{N_0} = ??$

A. $\log_2 M \frac{E_s}{N_0}$ B. $\frac{1}{\log_2 M} \frac{E_s}{N_0}$ C. $M \frac{E_s}{N_0}$ D. $\frac{1}{M} \frac{E_s}{N_0}$

Bit Error Rate (BER)

- Number of Constellation Points: M
- Bits per symbol: $\log_2 M$
- Signal-to-Noise Ratio: SNR
- SNR per symbol: $\frac{E_s}{N_0}$
- SNR per bit: $\frac{E_b}{N_0} = \frac{1}{\log_2 M} \frac{E_s}{N_0}$
- Symbol Error Rate: $SER = P_s$
- Bit Error Rate: $BER = P_b$

- Relation between P_s and P_b?

A. $P_b \le P_s$ B. $P_b = P_s$ C. $P_b > P_s$ D. Depends on Modulation Used

Bit Error Rate (BER)

- Number of Constellation Points: M
- Bits per symbol: $\log_2 M$
- $P_b \leq P_s$?

1 symbol error can at most generate $\log_2 M$ bit errors.

$$P_b = \frac{Number\ of\ Error\ Bits}{Total\ Number\ of\ Bits} \leq \frac{Number\ of\ Error\ Symbols \times \log_2 M}{Total\ Number\ of\ Symbols \times \log_2 M} = P_s$$

How much is P_b less than P_s?

Depends on the code used: How we assign bits to symbol

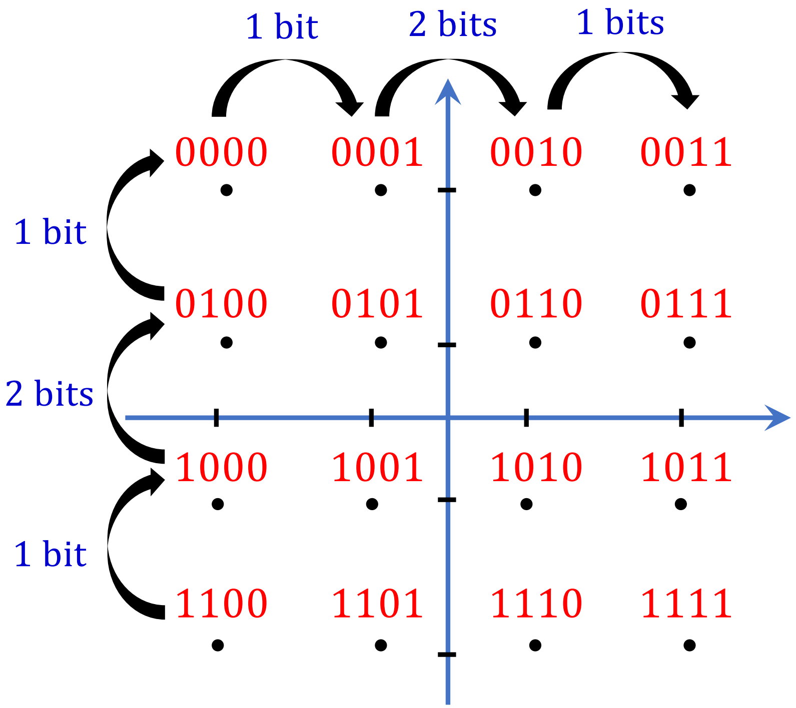
How much is P_b less than P_s?

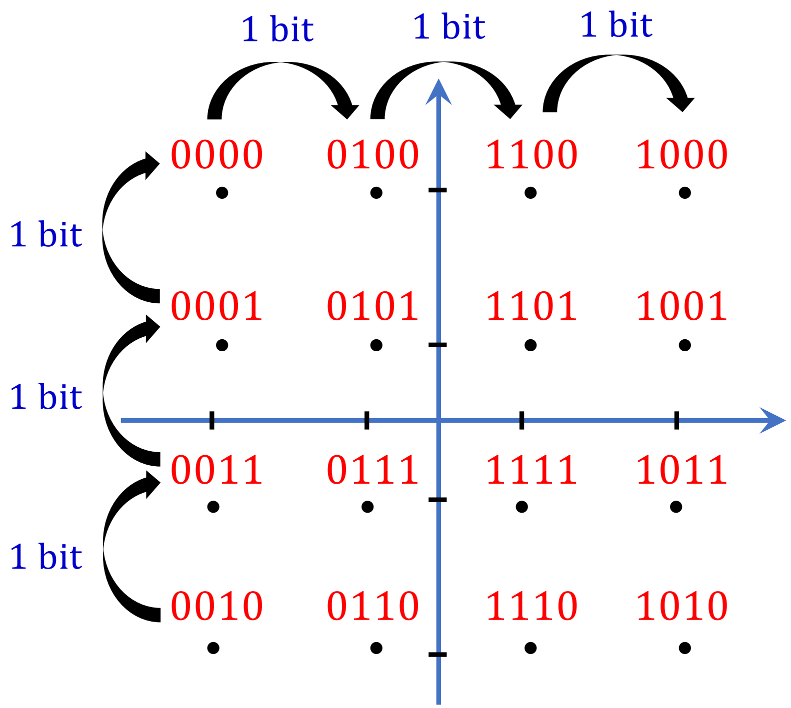
Depends on the code used: How we assign bits to symbol

$P_b \approx \frac{1}{3} P_s$

16-QAM

$P_b \approx \frac{1}{4} P_s$





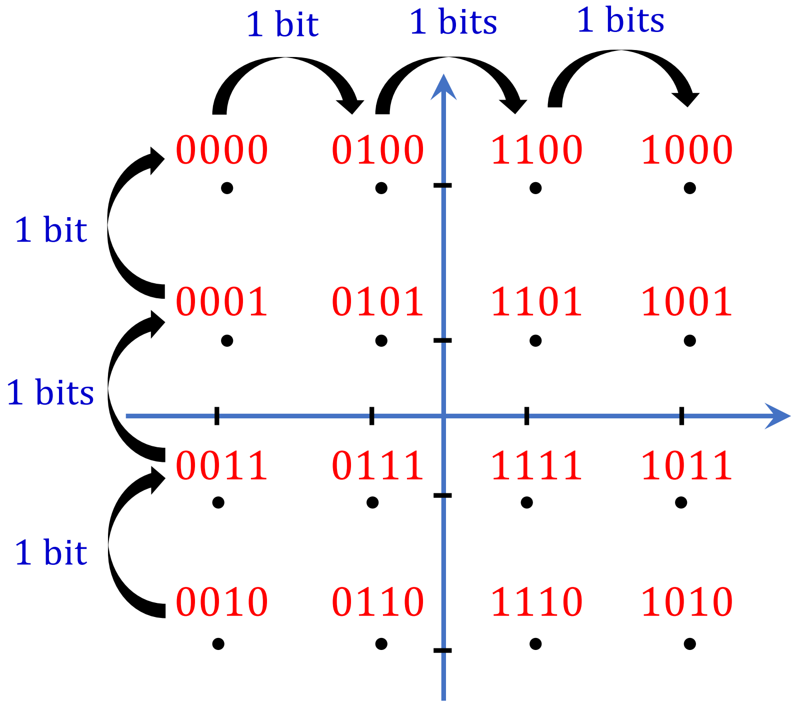
Most likely error occurs between nearest neighbor constellation points!

Minimize bit flips between nearest neighbors.

How much is P_b less than P_s?

Depends on the code used: How we assign bits to symbol

16-QAM $\quad P_b \approx \dfrac{1}{4} P_s$



Gray Codes

Bit flips between nearest neighbors $= 1$

$$P_b \approx \frac{1}{\log_2 M} P_s$$

Bit Error Rate (BER)

- Number of Constellation Points: M
- Bits per symbol: $\log_2 M$
- Assuming Gray Code is used:

$$BER:\ P_b = \frac{1}{\log_2 M} P_s$$

- Nearest Neighbor Approximation:

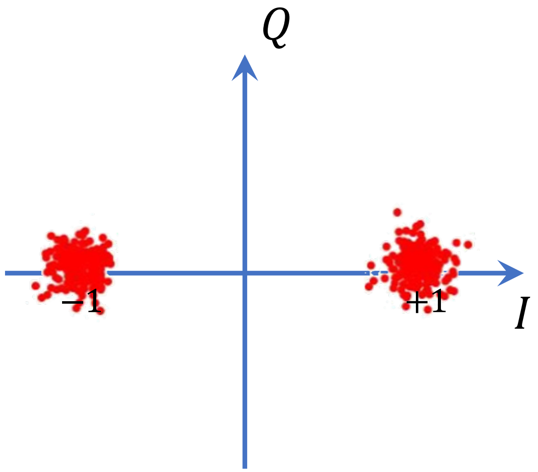
$$SER:\ P_s \approx \#\, nearest\ neighbors \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

Bit Error Rate (BER)

$$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = Q\left(\frac{2\sqrt{E_s}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{2E_s/N_0}\right)$$

$$BER:\ P_b = \frac{1}{\log_2 M} P_s = Q\left(\sqrt{2E_b/N_0}\right)$$

BPSK:



$$d_{min} = 2\sqrt{E_s}$$

$$\#nn = 1$$

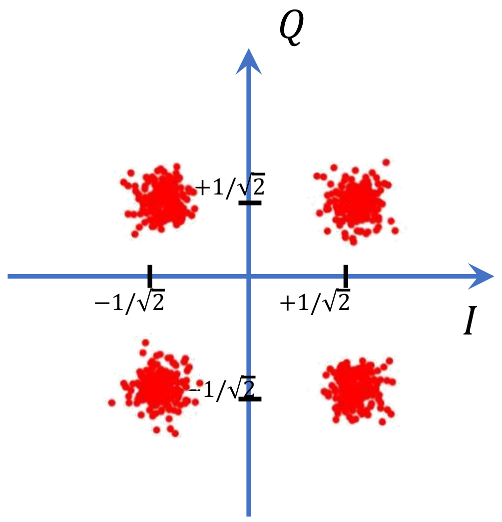
$$\log_2 M = 1$$

Bit Error Rate (BER)

$$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = 2Q\left(\sqrt{E_s/N_0}\right)$$

$$BER:\ P_b = \frac{1}{\log_2 M} P_s = Q\left(\sqrt{2E_b/N_0}\right)$$

4-QAM/QPSK



$$d_{min} = \frac{2}{\sqrt{2}}\sqrt{E_s}$$

$$\#nn = 2$$

$$\log_2 M = 2$$

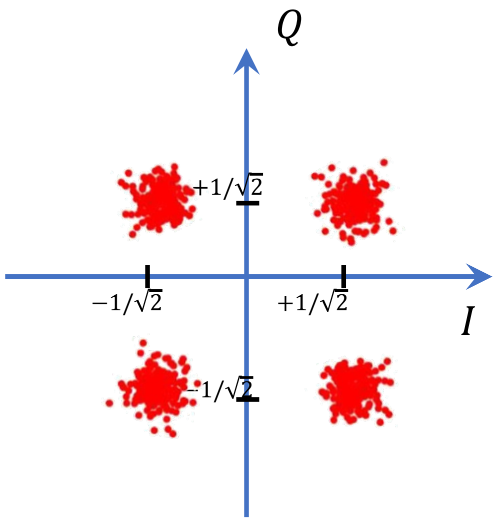
Bit Error Rate (BER)

$$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = 2Q\left(\sqrt{E_s/N_0}\right)$$

$$BER:\ P_b = \frac{1}{\log_2 M} P_s = Q\left(\sqrt{2E_b/N_0}\right)$$

Same as BPSK with 2× Bitrate!

4-QAM/QPSK



$$d_{min} = \frac{2}{\sqrt{2}}\sqrt{E_s}$$

$$\#nn = 2$$

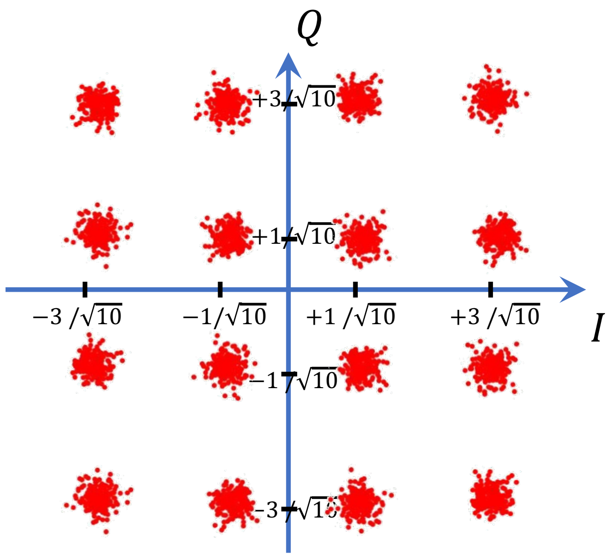
$$\log_2 M = 2$$

Bit Error Rate (BER)

$$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = 4Q\left(\sqrt{E_s/5N_0}\right)$$

$$BER:\ P_b = \frac{1}{\log_2 M} P_s = Q\left(\sqrt{4E_b/5N_0}\right)$$

16-QAM



$$d_{min} = \frac{2}{\sqrt{10}}\sqrt{E_s}$$

$$\#nn = 4$$

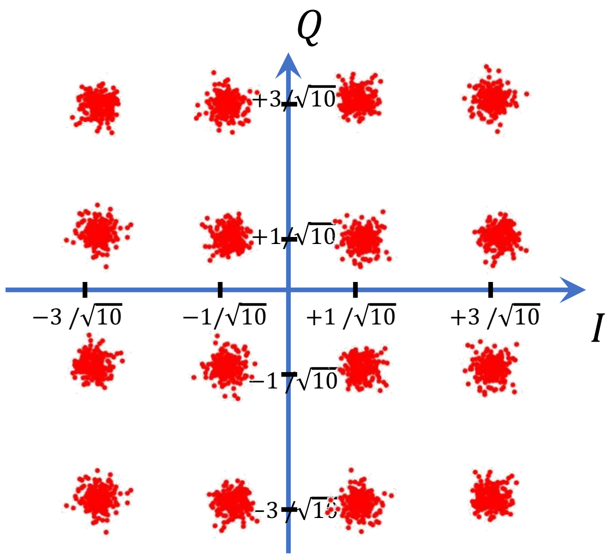
$$\log_2 M = 4$$

Bit Error Rate (BER)

$$SER\text{: } P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = 4Q\left(\sqrt{\frac{3E_s/N_0}{M-1}}\right)$$

$$BER\text{: } P_b = \frac{1}{\log_2 M} P_s = \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\log_2 M\, E_b/N_0}{M-1}}\right)$$

M-QAM



$$d_{min} = \sqrt{\frac{6E_s}{M-1}}$$

$$\#nn = 4$$

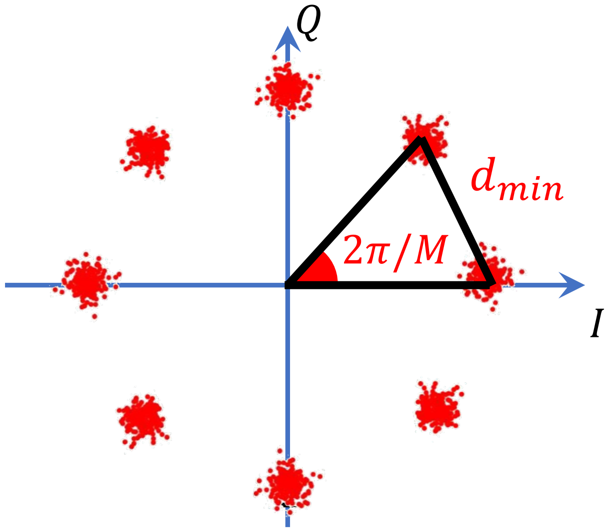
$$\log_2 M$$

Bit Error Rate (BER)

$$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \quad = 2Q\left(\sqrt{2E_s/N_0}\sin\left(\frac{\pi}{M}\right)\right)$$

$$BER:\ P_b = \frac{1}{\log_2 M} P_s \quad = \frac{2}{\log_2 M} Q\left(\sqrt{2\log_2 M\, E_b/N_0}\sin\left(\frac{\pi}{M}\right)\right)$$

M-PSK



$$d_{min} = 2\sin\left(\frac{\pi}{M}\right)\sqrt{E_s}$$

$$\#nn = 2$$

$$\log_2 M$$

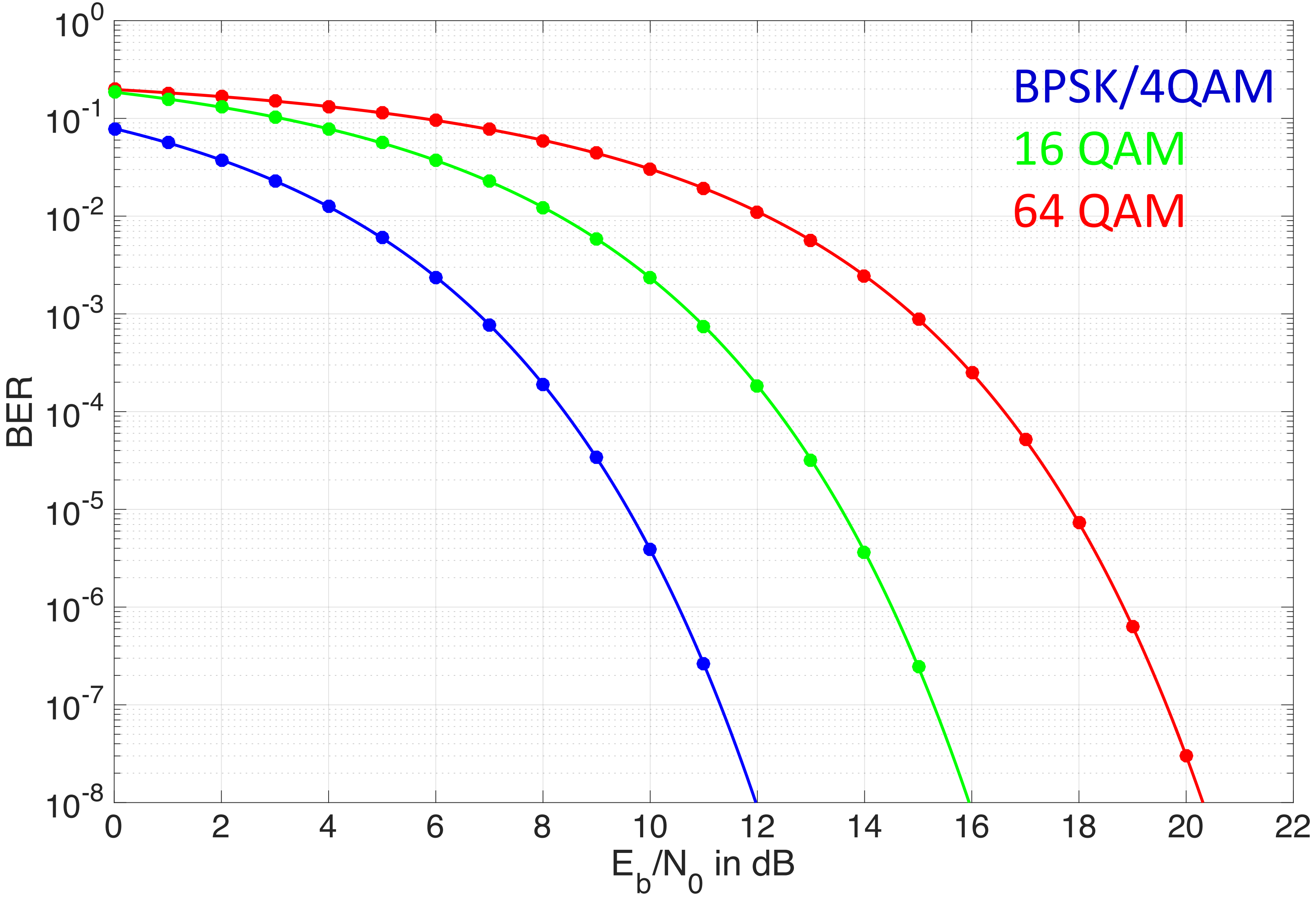
Bit Error Rate (BER)

	$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$	$BER:\ P_b = \frac{1}{\log_2 M} P_s$
• BPSK	$Q(\sqrt{2E_s/N_0})$	$Q(\sqrt{2E_b/N_0})$
• QPSK/4-QAM	$2\,Q(\sqrt{E_s/N_0})$	$Q(\sqrt{2E_b/N_0})$
• MPAM	$\frac{2(M-1)}{M}\,Q\left(\sqrt{\frac{6E_s/N_0}{M^2-1}}\right)$	$\frac{2(M-1)}{M\log_2 M}\,Q\left(\sqrt{\frac{6\log_2 M\,E_b/N_0}{M^2-1}}\right)$
• MPSK	$2\,Q\left(\sqrt{2E_s/N_0}\sin\left(\frac{\pi}{M}\right)\right)$	$\frac{2}{\log_2 M}\,Q\left(\sqrt{2\log_2 M\,E_b/N_0}\sin\left(\frac{\pi}{M}\right)\right)$
• MQAM	$4\,Q\left(\sqrt{\frac{3E_s/N_0}{M-1}}\right)$	$\frac{4}{\log_2 M}\,Q\left(\sqrt{\frac{3\log_2 M\,E_b/N_0}{M-1}}\right)$

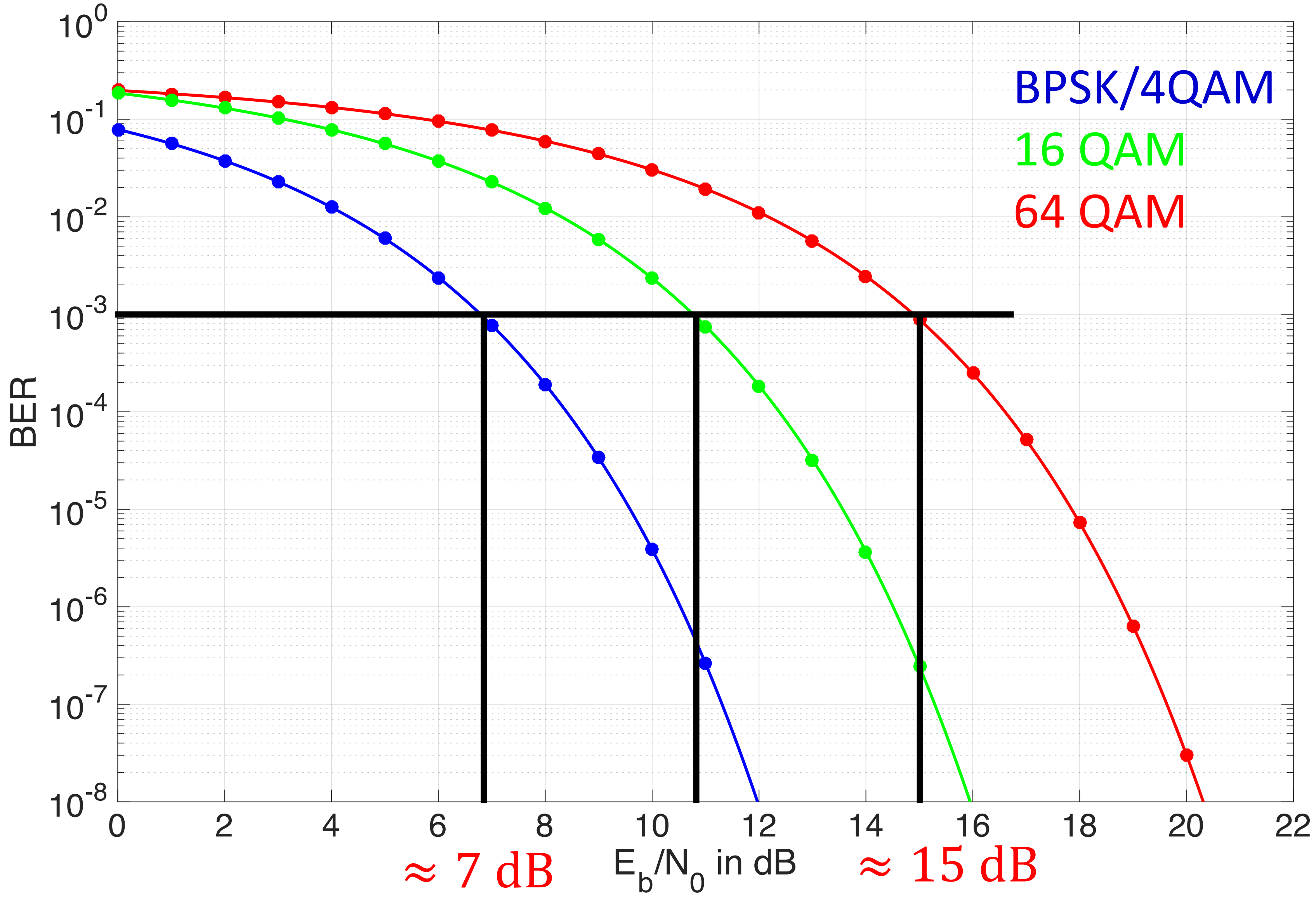
$\alpha_M Q(\sqrt{\beta_M E_s/N_0})$ $\quad\quad$ $\hat{\alpha}_M Q\left(\sqrt{\hat{\beta}_M E_b/N_0}\right)$

$$\hat{\alpha}_M = \frac{\alpha_M}{\log_2 M}, \qquad \hat{\beta}_M = \beta_M \log_2 M\,, \qquad \frac{E_b}{N_0} = \frac{1}{\log_2 M}\frac{E_s}{N_0}$$

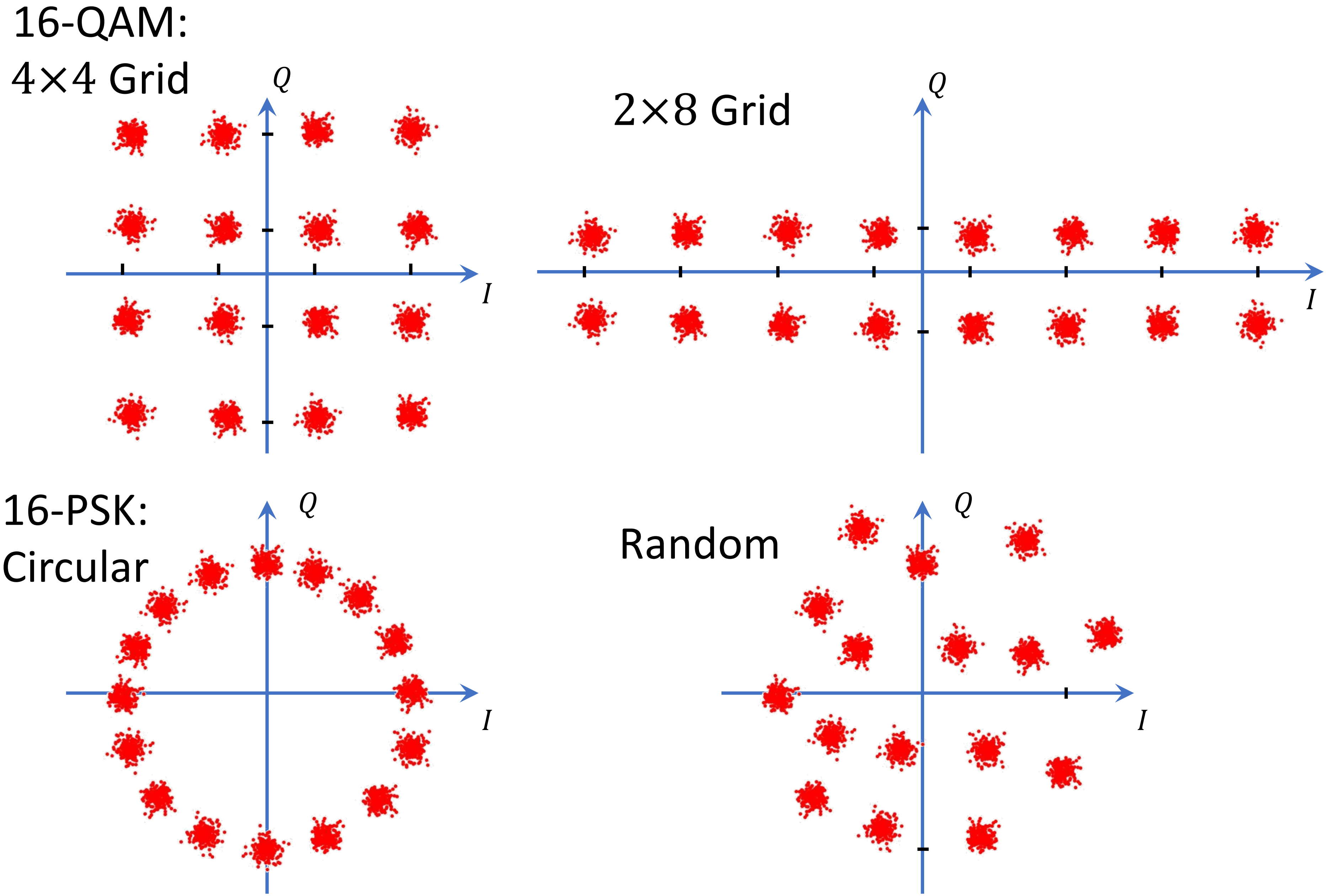
Bit Error Rate (BER)



Bit Error Rate (BER)



Shape of the Constellation (e.g. $M = 16$)

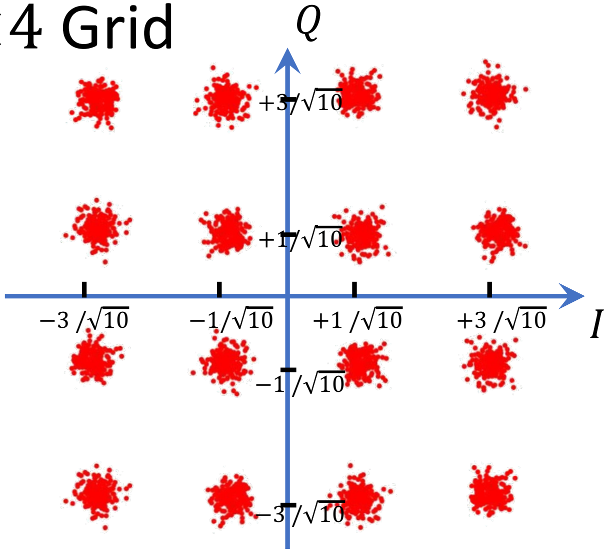


Shape of the Constellation (e.g. $M = 16$)

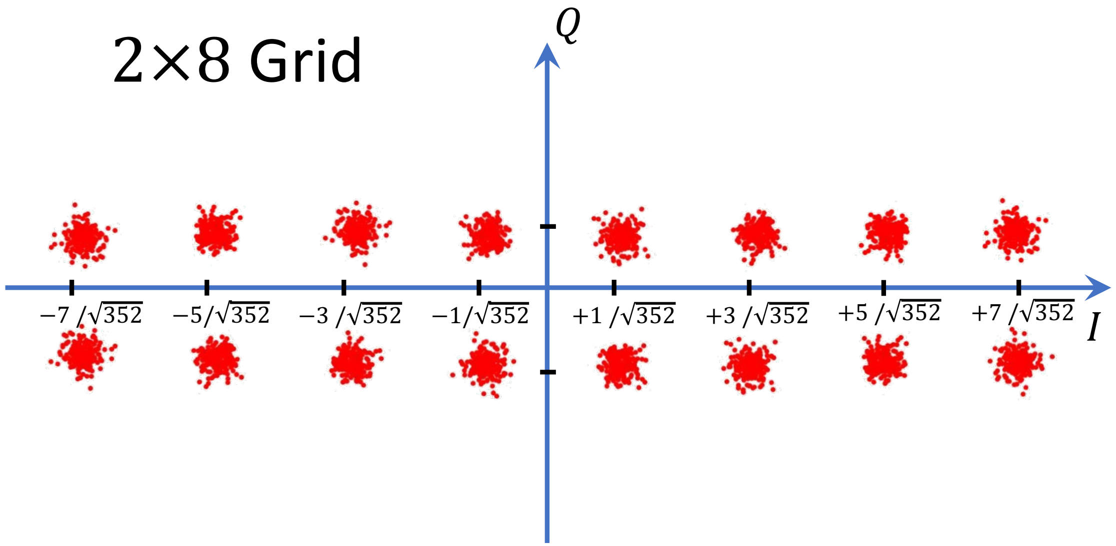
$$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

Goal: Maximize d_{min} for the same overall power

4×4 Grid



2×8 Grid



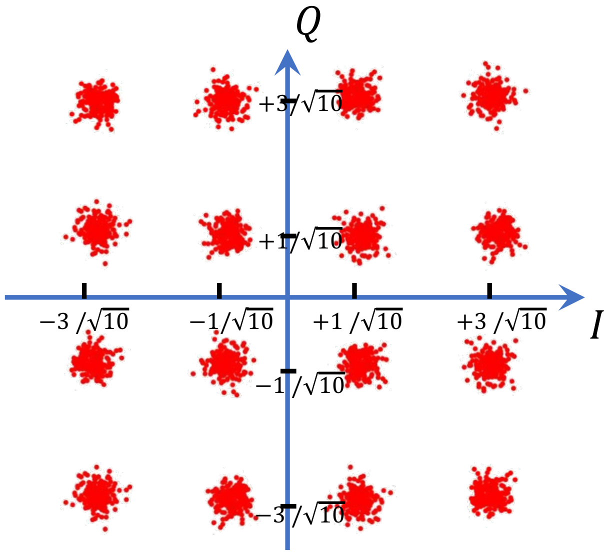
$$d_{min} = \frac{2}{\sqrt{10}}\sqrt{E_s} \quad \gg \quad d_{min} = \frac{2}{\sqrt{352}}\sqrt{E_s}$$

Shape of the Constellation (e.g. $M = 16$)

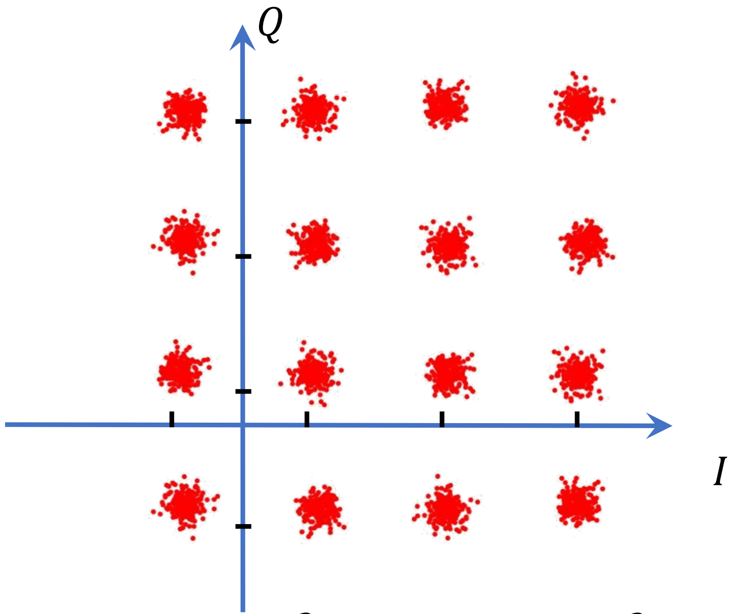
$$SER: P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

Goal: Maximize d_{min} for the same overall power

Average of constellation should be zero.



$E[|x(t)|^2] = 1$



$E[|x(t)|^2] = 1 + |E[x(t)]|^2$

If Average is not zero, we have higher TX Power for the same d_{min} or smaller d_{min} for same TX power.

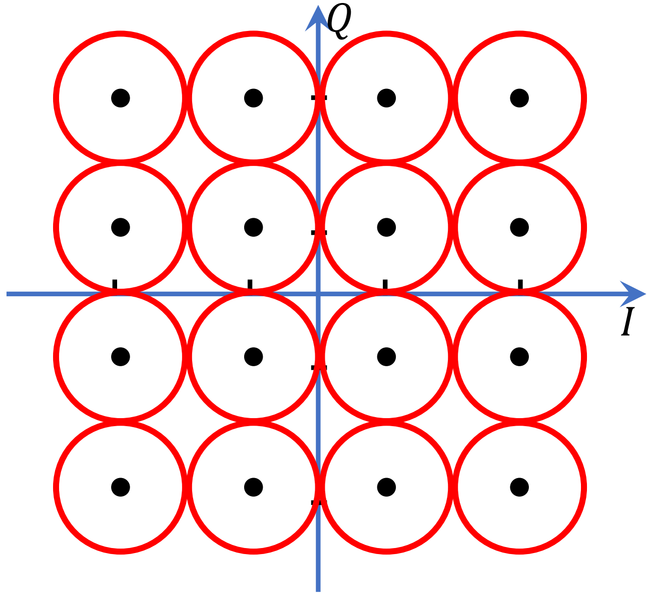
Shape of the Constellation (e.g. $M = 16$)

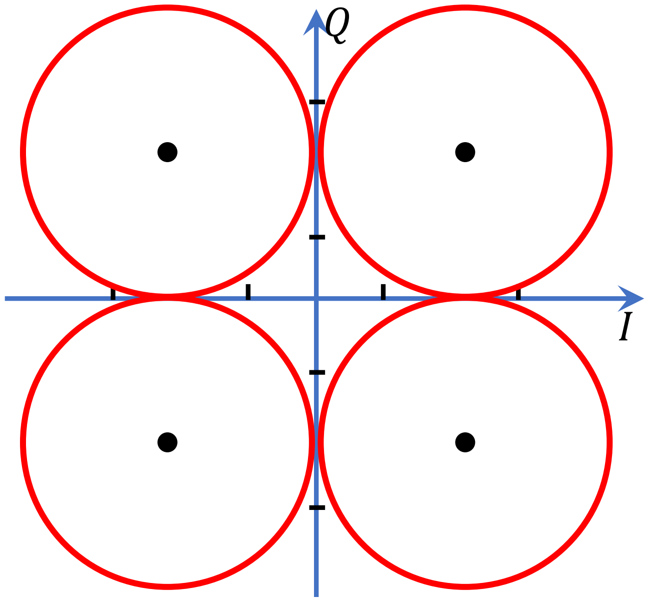
$$SER: P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

Goal: Maximize d_{min} for the same overall power

Average of constellation should be zero.

Geometric packing problem.





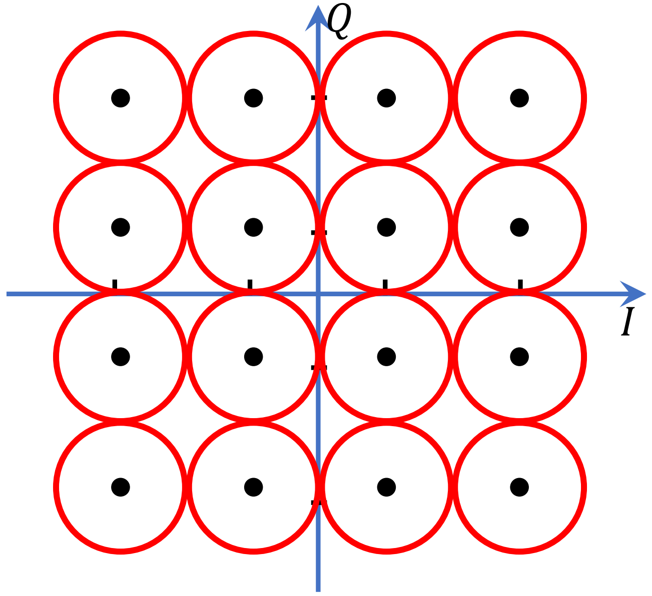
Shape of the Constellation (e.g. $M = 16$)

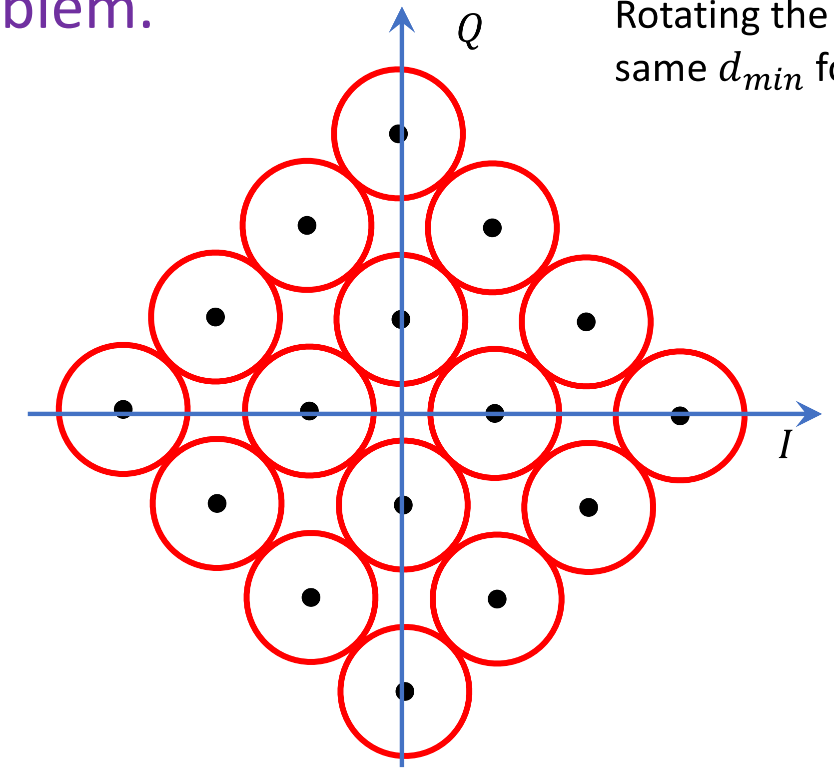
$$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

Goal: Maximize d_{min} for the same overall power

Average of constellation should be zero.

Geometric packing problem.





Rotating the constellation maintains same d_{min} for same overall power

Shape of the Constellation (e.g. $M = 16$)

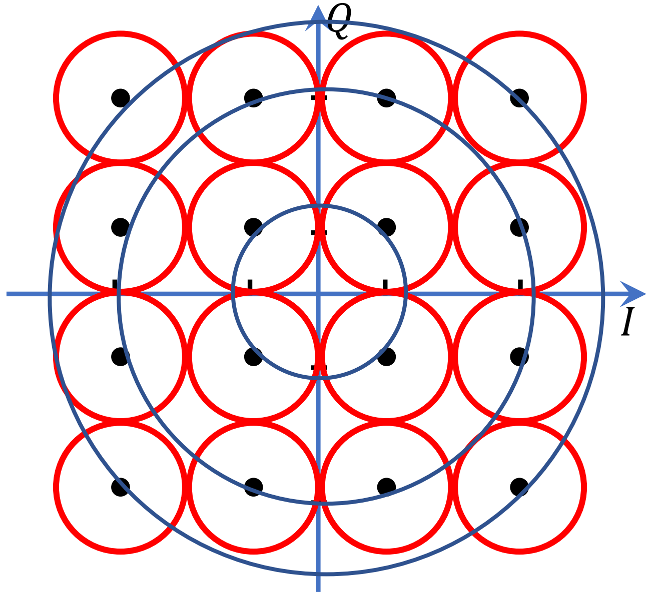
$$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

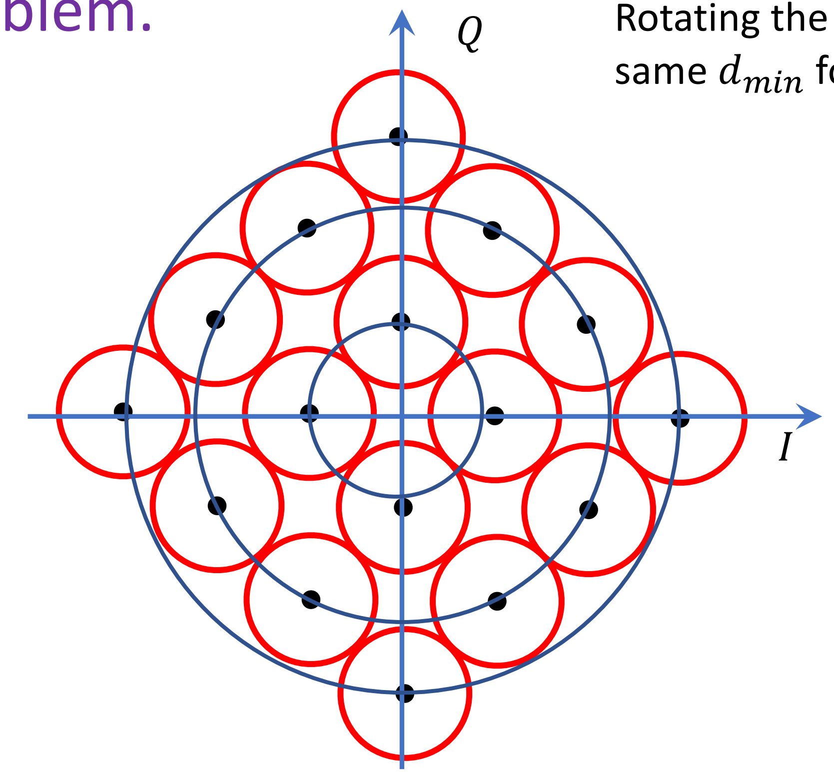
Goal: Maximize d_{min} for the same overall power

Average of constellation should be zero.

Geometric packing problem.

Rotating the constellation maintains same d_{min} for same overall power





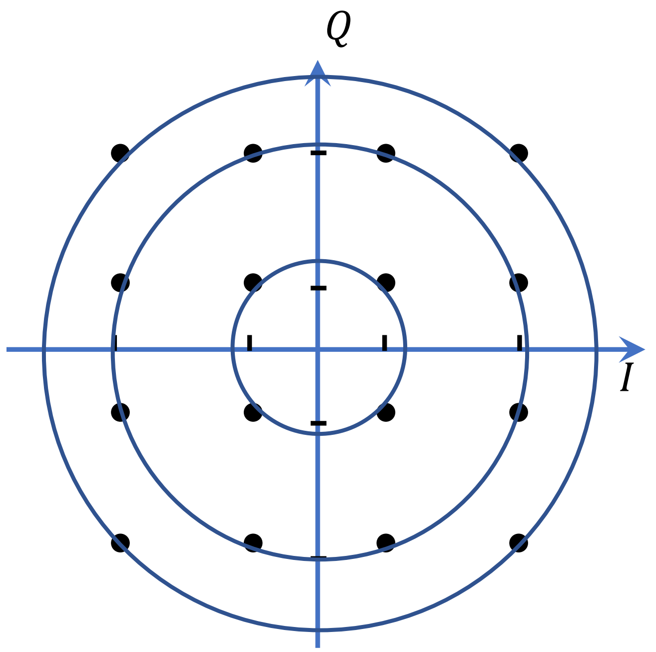
Shape of the Constellation (e.g. $M = 16$)

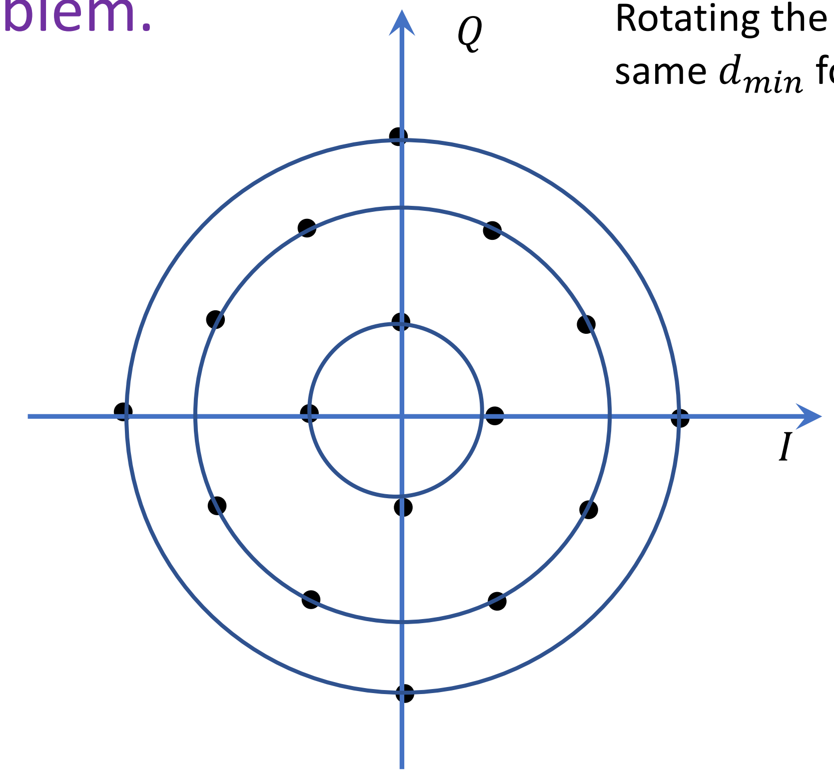
$$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

Goal: Maximize d_{min} for the same overall power

Average of constellation should be zero.

Geometric packing problem.





Rotating the constellation maintains same d_{min} for same overall power

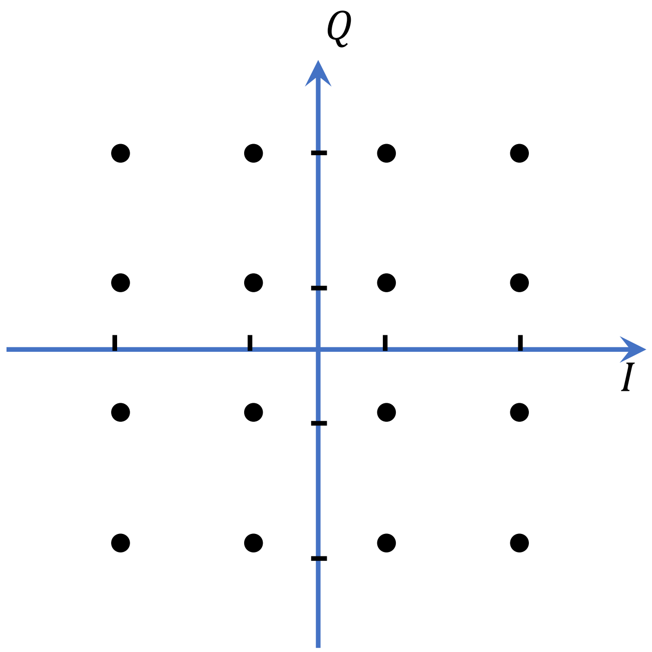
Shape of the Constellation (e.g. $M = 16$)

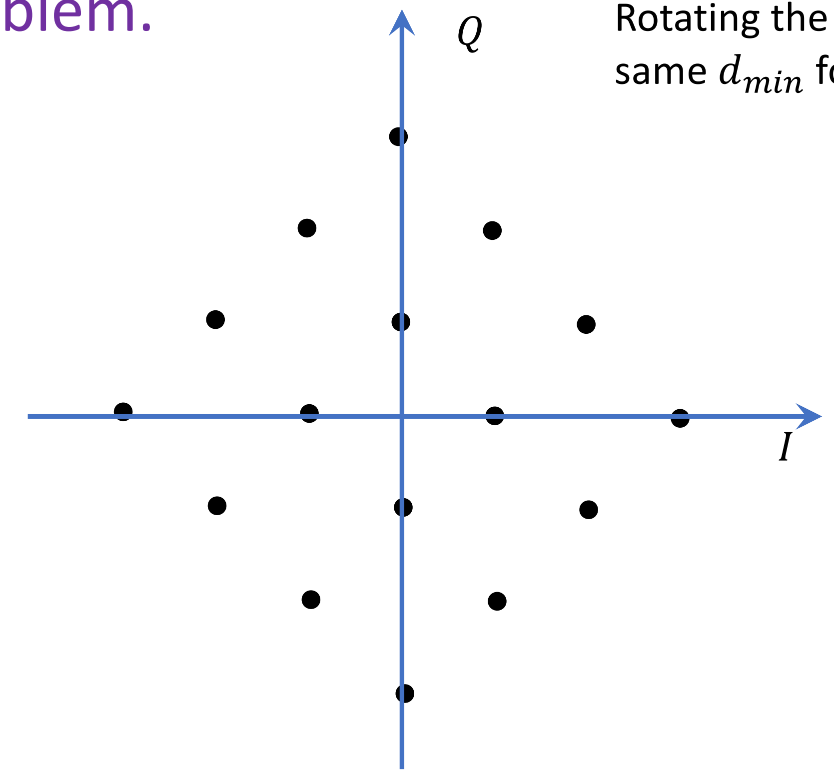
$$SER:\ P_s \approx \#nn \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

Goal: Maximize d_{min} for the same overall power

Average of constellation should be zero.

Geometric packing problem.





Rotating the constellation maintains same d_{min} for same overall power

QAM Is Popular!

- WiFi

  802.11n: BPSK, QPSK, 16-QAM, 64-QAM

  802.11ac: BPSK, QPSK, 16-QAM, 64-QAM, 256-QAM

  802.11ax (2019): BPSK, QPSK, 16-QAM, 64-QAM, 256-QAM, 1024-QAM

- LTE: QPSK, 16-QAM, 64-QAM

- Digital TV:

  DVB-C: 16-QAM, 64-QAM, 256-QAM

  DVB-C2: 16-QAM, 64-QAM, 256-QAM, 1024-QAM, 4096-QAM

  Next Gen. to include 16364-QAM & 65536-QAM

- Ethernet, Phone lines, Power lines…: 1024-QAM, 4096-QAM

- ADSL: 32768-QAM

65536-QAM

- Number of Constellation Points: $M = 65536$
- Bits per symbol: $\log_2 M = 16$
- QAM Grid: 256×256
- Need 8 bits to represent I and 8 bits to represent Q

What is the problem?

ADCs need to have a large dynamic range!

Quantization noise kicks in!

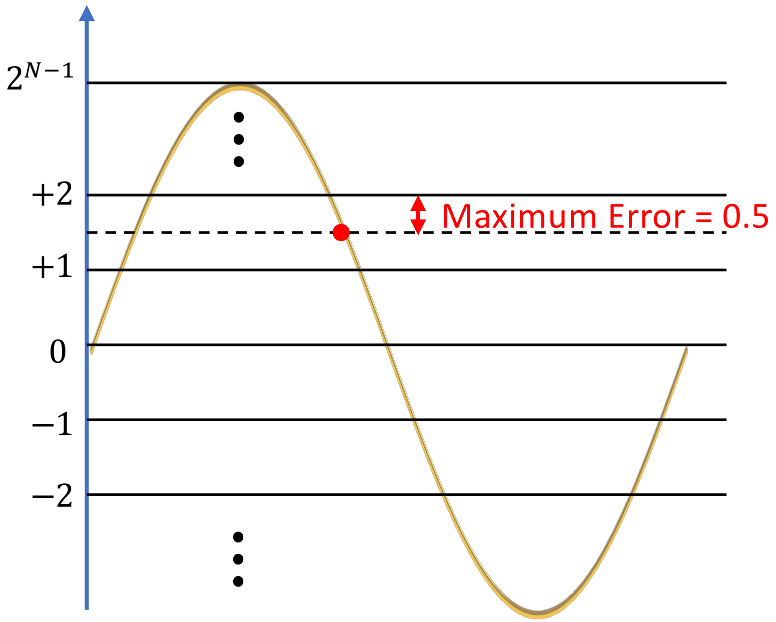
65536-QAM

- Number of Constellation Points: $M = 65536$
- Bits per symbol: $\log_2 M = 16$
- QAM Grid: 256×256
- Need 8 bits to represent I and 8 bits to represent Q
- ADC with K bits typically supports $N < K$ bit samples

ENOB: Effective Number of Bits $N < K$

Quantization Noise

- Consider N bit quantization



$$SNR_{quant.} = \frac{(Signal\ Amplitude)^2}{(Quantization\ Noise)^2}$$

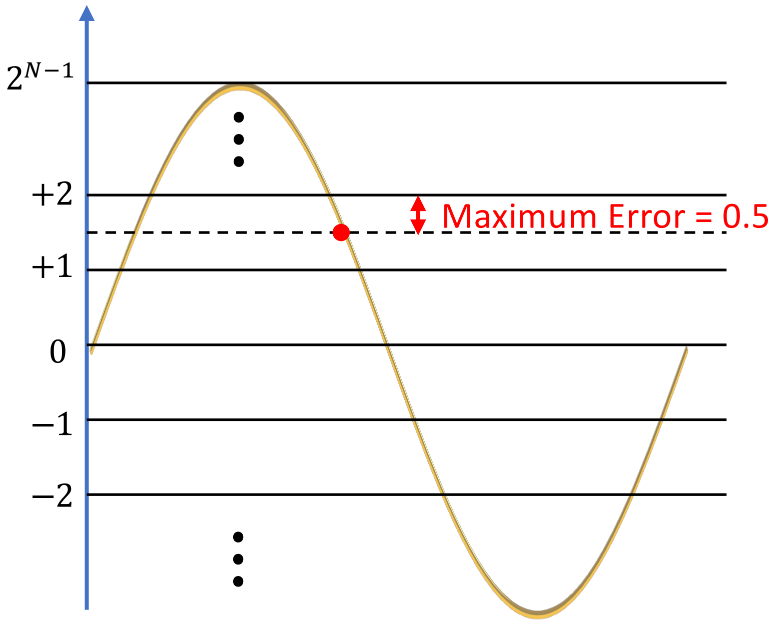
$$= \frac{(2^{N-1})^2}{(0.5)^2} = \frac{2^{2N-2}}{2^{-2}} = 2^{2N}$$

$$SNR_{quant.}(\mathrm{dB}) = 10\log_{10} 2^{2N} = 20N\log_{10} 2 \ = 6.02N$$

$\approx$ 6 dB per bit

Quantization Noise

- Consider N bit quantization



$$SNR_{quant.} = \frac{(Signal\ Amplitude)^2}{(Quantization\ Noise)^2}$$

$$= \frac{(2^{N-1})^2}{(0.5)^2} = \frac{2^{2N-2}}{2^{-2}} = 2^{2N}$$

$$SNR_{quant.}(\text{dB}) = 10\log_{10} 2^{2N} = 20N\log_{10} 2 = 6.02N$$

$\approx$ 6 dB per bit

Need: $SNR_{quant.} > SNR_{thermal}$

65536-QAM: $BER = \frac{1}{4} Q\left(\sqrt{\frac{3\ SNR}{65535}}\right)$ ➡ Need: $SNR > 52$ dB to achieve $BER < 10^{-3}$

➡ Need: $N > 9$ bits

65536-QAM

- Number of Constellation Points: $M = 65536$
- Bits per symbol: $\log_2 M = 16$
- QAM Grid: 256×256
- Need 8 bits to represent I and 8 bits to represent Q
- ADC with K bits typically supports $N < K$ bit samples

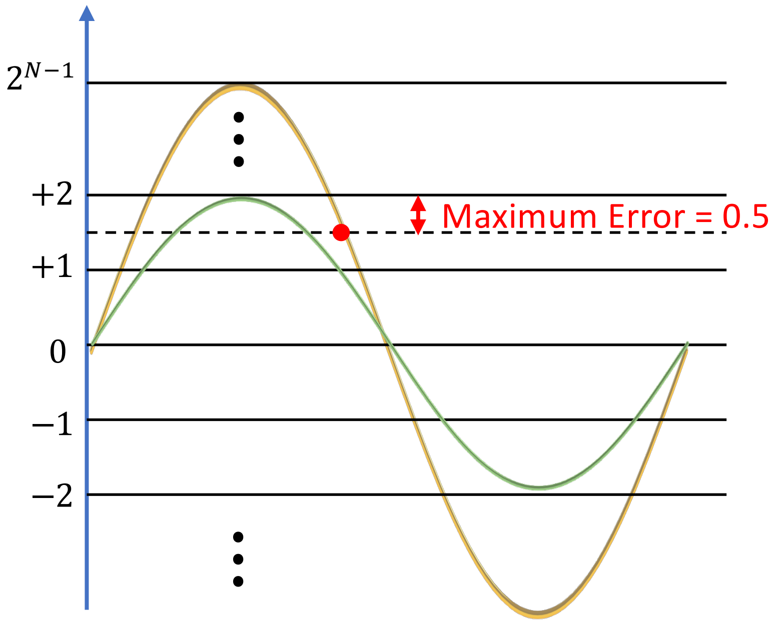
ENOB: Effective Number of Bits $N < K$

- Need very high SNR to achieve reasonable BER.
- Need to minimize quantization noise: $N \geq 12$

Need at least a 14 bit or 16 bit ADC

Quantization Noise

- Consider N bit quantization



$$SNR_{quant.} = \frac{(Signal\ Amplitude)^2}{(Quantization\ Noise)^2}$$

$$= \frac{(2^{N-1})^2}{(0.5)^2} = \frac{2^{2N-2}}{2^{-2}} = 2^{2N}$$

$$SNR_{quant.}(\text{dB}) = 10\log_{10} 2^{2N} = 20N\log_{10} 2 = 6.02N$$

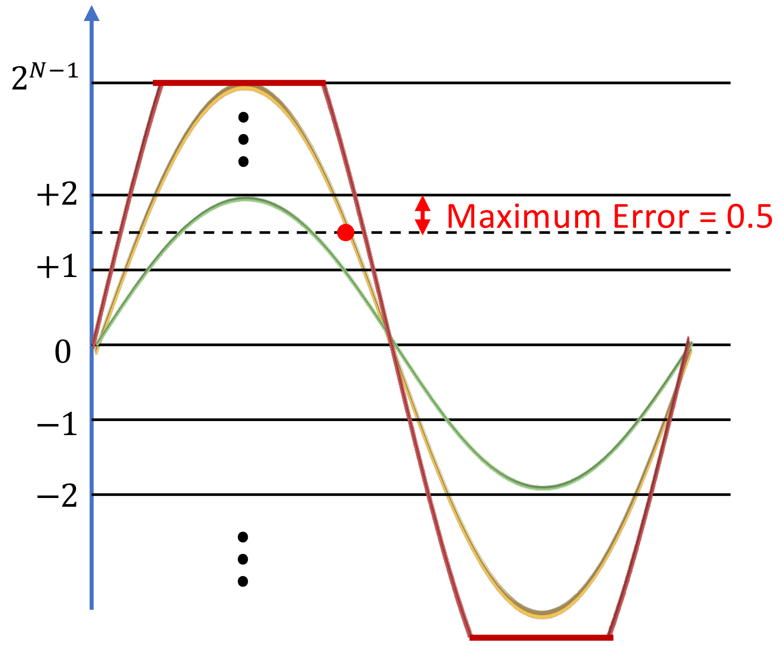
$\approx$ 6 dB per bit

Assumes signal amplitude fills maximum quantization level.

Signals arrives too attenuated → fill smaller quantization levels → Low $SNR_{quant.}$

Quantization Noise

- Consider N bit quantization



$$SNR_{quant.} = \frac{(Signal\ Amplitude)^2}{(Quantization\ Noise)^2}$$

$$= \frac{(2^{N-1})^2}{(0.5)^2} = \frac{2^{2N-2}}{2^{-2}} = 2^{2N}$$

$$SNR_{quant.}(\text{dB}) = 10\log_{10} 2^{2N} = 20N\log_{10} 2 = 6.02N$$

≈ 6 dB per bit

Assumes signal amplitude fills maximum quantization level.

Signals arrives too attenuated → fill smaller quantization levels → Low $SNR_{quant.}$

Signals arrives too amplified → Clipping

AGC: Automatic Gain Control

- Consists of Variable Gain Amplifier and Control Circuit
- Adjust the gain to minimize quantization noise & avoid clipping.
- Receiver Circuit: