

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 15

Midterm

Information Theory and Coding

Nov. 02, 2021

3 problems, 48 points

165 minutes

1 sheet (2 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

PROBLEM 1. (16 points)

- (a) (2 pts) Suppose A, B, C are random variables. For any function g show that $I(A; g(B)|C) \leq I(A; B|C)$.

Suppose $Z, U_1, U_2, \dots$ are random variables taking values in finite alphabets $\mathcal{Z}, \mathcal{U}_1, \mathcal{U}_2, \dots$.

- (b) (4 pts) Let $\Delta_i = I(Z; U_i | U^{i-1})$. Show that $\lim_i \Delta_i = 0$.

[Hint: consider $s_n := \sum_{i=1}^n \Delta_i$.]

- (c) (2 pts) Suppose that Z_i is a Markov process, and $U_i = f_i(Z_i)$ are functions of the Z_i 's. Observe that for $i > 1$, $I(Z_1; Z_2, \dots, Z_i | Z_2) = 0$. Show that $I(Z_1; U_{i+1} | Z_2, U_2, \dots, U_i) = 0$ for $i > 1$.

[Hint: first show $I(Z_1; U_2, \dots, U_{i+1} | Z_2) = 0$.]

- (d) (4 pts) Suppose Z_i is a *stationary* Markov process with a finite alphabet, and $U_i = f(Z_i)$ (Note that the same function f is used for each i to find U_i). Show that $a_i = H(U_i | Z_1, U^{i-1})$ is non-decreasing.
- (e) (4 pts) Continuing with (d) show that a_i converges to the entropy rate of the process $(U_1, U_2, \dots)$.

PROBLEM 2. (10 points) For $U^k = (U_1, \dots, U_k)$, let V_i denote all the U 's except U_i ; e.g., $V_1 = (U_2, \dots, U_k)$, $V_k = U^{k-1}$.

(a) (4 pts) Show that $\sum_{i=1}^k H(V_i) \geq (k-1)H(U^k)$.

[Hint: for any i , $H(V_i) + H(U_i|V_i) = H(U^k)$.]

For $U^n = (U_1, \dots, U_n)$ and $\mathcal{S}$ a subset of the indices $\{1, \dots, n\}$, let $U_{\mathcal{S}}$ denote $(U_i : i \in \mathcal{S})$, the collection of U_i 's with i in $\mathcal{S}$. Let $H_k = \sum_{\mathcal{S}:|\mathcal{S}|=k} H(U_{\mathcal{S}})$. (E.g., $H_1 = H(U_1) + \dots + H(U_n)$; $H_n = H(U^n)$).

(b) (4 pts) Show that $(n-k)H_k \geq kH_{k+1}$.

[Hint: From (a) we know that for any $\mathcal{S}$ with $|\mathcal{S}| = k+1$, $\sum_{\mathcal{T}:|\mathcal{T}|=k} \mathbb{1}\{\mathcal{T} \subset \mathcal{S}\} H(U_{\mathcal{T}}) \geq kH(U_{\mathcal{S}})$, where the sum over $\mathcal{T}$ is over all subsets of $\{1, \dots, n\}$ with $|\mathcal{T}| = k$.]

(c) (2 pts) Show that $\frac{1}{k}H_k / \binom{n}{k} \geq \frac{1}{k+1}H_{k+1} / \binom{n}{k+1}$.

PROBLEM 3. (22 points) As the Lempel-Ziv (LZ) algorithm reads the sequence $u_1, u_2 \dots$ from an alphabet $\mathcal{U}$, it produces the bit-representations of the words $w_1, w_2, \dots$ it has parsed the input sequence into. Let D_m be the dictionary of the LZ algorithm as w_{m+1} is being parsed. Let l_m be the length of the longest word in D_m .

- (a) (2 pts) Suppose $u_1 u_2 \dots$ equals $aaaaa \dots$. Find the sequence l_m .
- (b) (4 pts) Give an upper bound on l_m in terms of m that holds for all $u_1 u_2 \dots$.

Suppose u_{n+1} is the first letter of w_{m+1} . The representation of w_{m+1} is emitted by the LZ algorithm only when the entire word w_{m+1} is formed; consequently the recipient of the bits will have to wait until then to learn u_{n+1} .

- (c) (4 pts) Show that the recipient of the bit descriptions is guaranteed to reconstruct u_{n+1} before the LZ encoder has read u_{n+m^*} where $m^* = m^*(u^n)$, the number of words in a maximal distinct parsing.

By (c) an upper bound on $m^*(u^n)$ gives us an idea about the delay between the reading of u_{n+1} by LZ and its reconstruction from the bit-stream emitted by LZ. We proceed to find such an upper bound. To avoid trivialities assume $A = |\mathcal{U}| \geq 2$.

- (d) (4 pts) Let $w_1, \dots, w_m$ be a distinct parsing of u^n . Let k satisfy $1 + A + \dots + A^{k-1} \leq \frac{m}{2}$. Show that $n \geq \frac{1}{2}mk$.

- (e) (4 pts) Show that
$$n \geq \frac{1}{2}m^*(u^n) \lfloor \log_A \left(1 + (A-1) \frac{m^*(u^n)}{2} \right) \rfloor$$
$$\geq \frac{1}{2}m^*(u^n) \log_A \frac{m^*(u^n)}{2A}.$$

[Hint: $1 + A + A^2 + \dots + A^{k-1} = \frac{A^k - 1}{A - 1}$.]

- (f) (4 pts) Show that $m^*(u^n) \leq 4n / \log_A(n)$ for sufficiently large n .

[Hint: Either $m < 2A\sqrt{n}$ or $m \geq 2A\sqrt{n}$. In the first case $m \leq 4n / \log_A(n)$ for n sufficiently large since $\sqrt{n} \log n$ grows slower than n . In the second case ...]