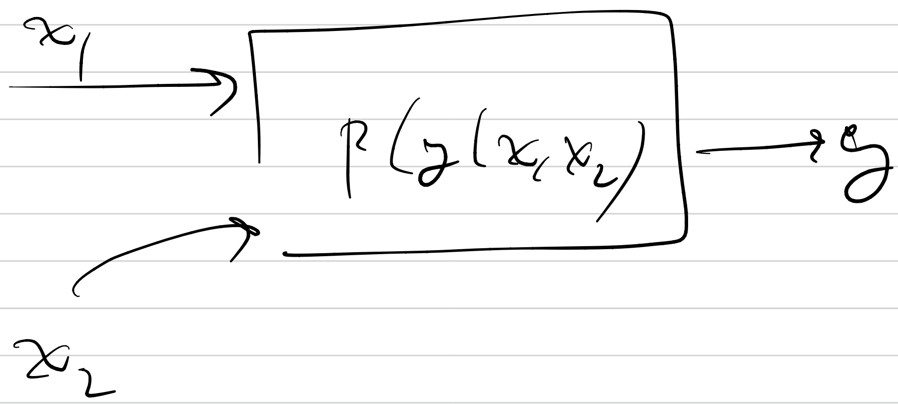
Information Theory & Coding

Dec. 15th, 2020

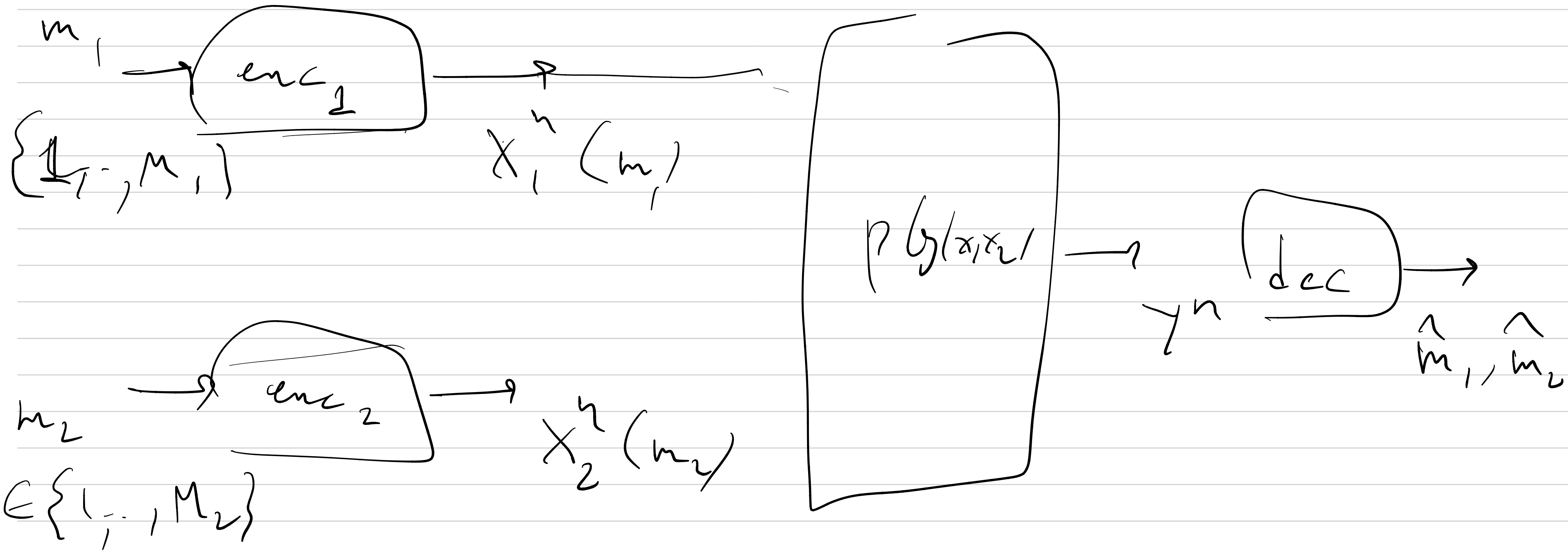
Multiple-Access Channel (2 senders)

- We are given a "channel" with 2 sender & 1 receiver



Discrete memoryless. We want to design

enc$_1$, enc$_2$, dec :



$$R_1 = \frac{1}{n} \log M_1$$

$$R_2 = \frac{1}{n} \log M_2$$

$P_e = \Pr\big((\hat{m}_1, \hat{m}_2) \neq (m_1, m_2)\big)$

(assume m_1, m_2 are uniformly distributed, indep) for a given channel

Def: we will say (R_1, R_2) is achievable if

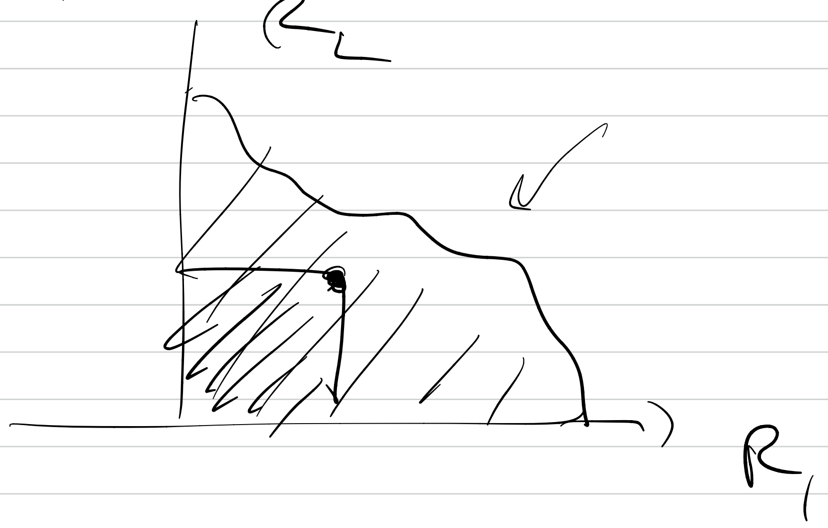
$\forall \varepsilon > 0$, $\exists$ enc_1, enc_2, dec s.t

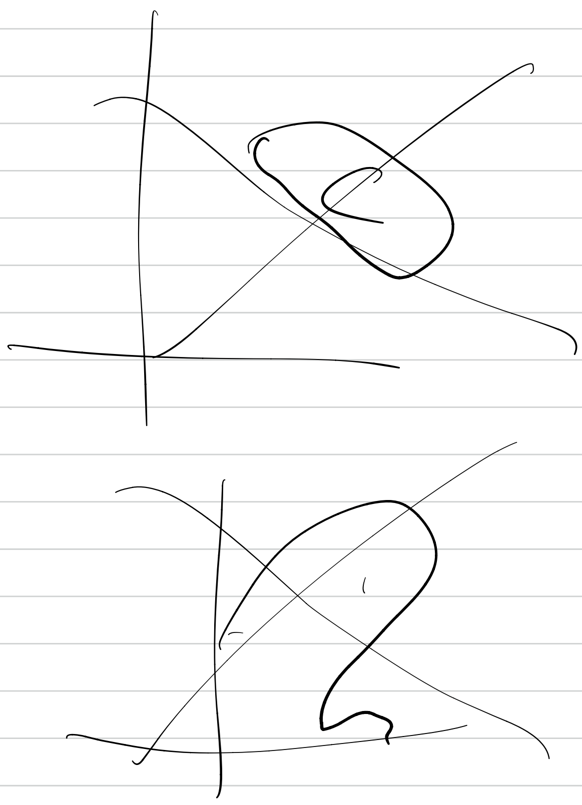
$\text{rate}(enc_1) \geq R_1$

$\text{rate}(enc_2) \geq R_2$

$P_{error} < \varepsilon$

let $C(\text{channel}) = \{ (R_1, R_2) : \text{achievable} \}$





$C(\text{channel})$ has to be convex subset of $\mathbb{R}^2$:

if $(R_1', R_2') \in C$ and $(R_1'', R_2'') \in C$

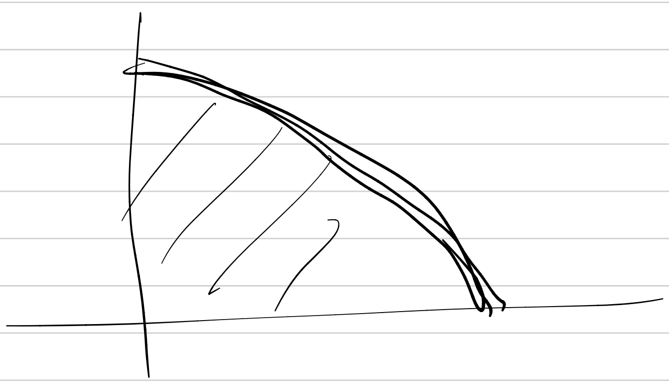
then $(R_1, R_2) = \lambda(R_1', R_2') + (1-\lambda)(R_1'', R_2'')$ $\quad 0 \leq \lambda \leq 1$

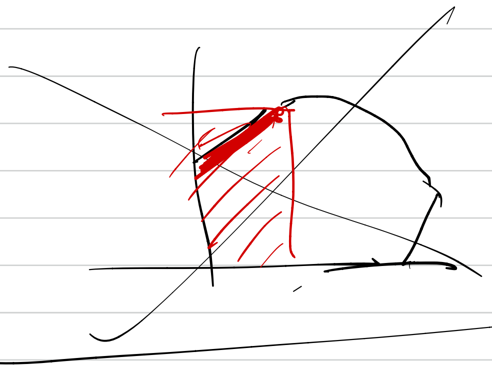
can also be achieved by using

(enc_1', enc_2', dec') a fraction λ of the time

(" " ") " " $(1-\lambda)$ " "

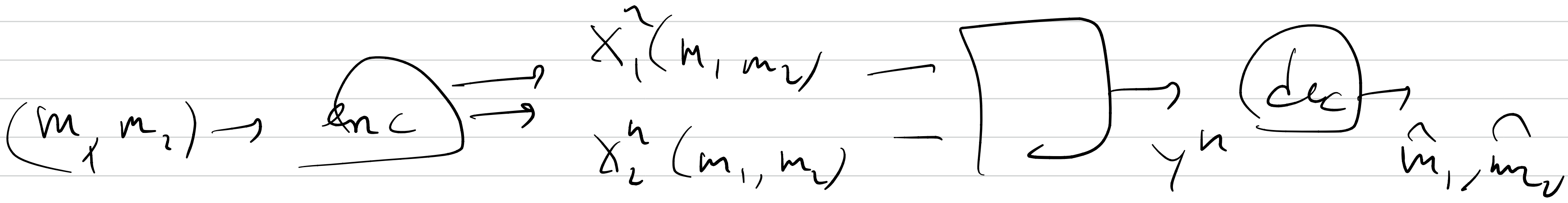
⇒ C has to be look like

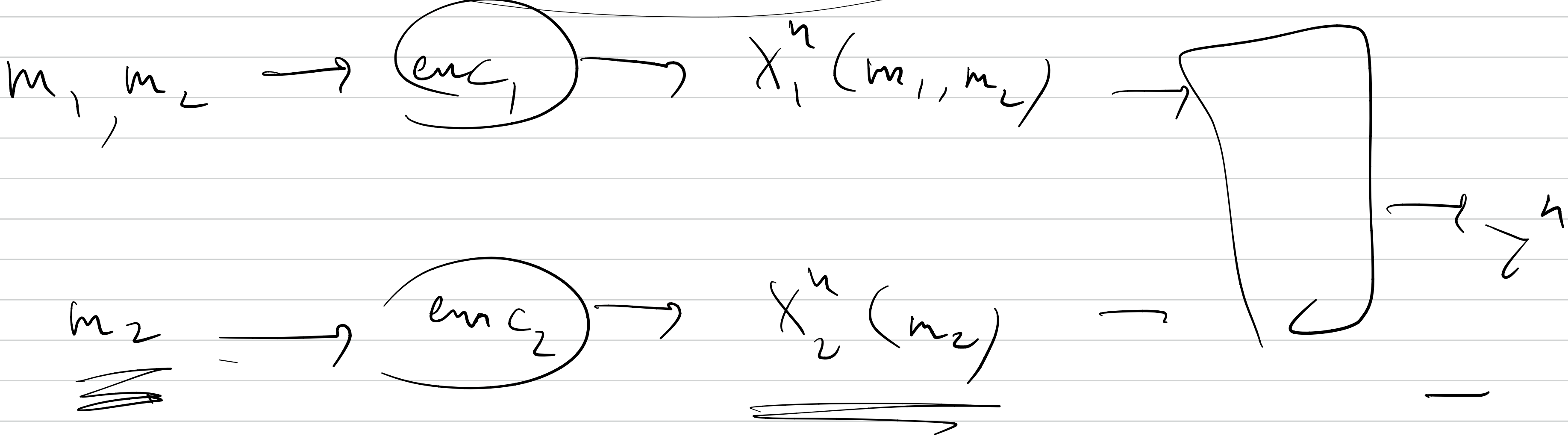




$(R_1, R_2) \in C \Rightarrow (\underset{\underset{R_1}{\wedge}}{R_1'}, \underset{\underset{R_2}{\wedge}}{R_2'}) \in C$

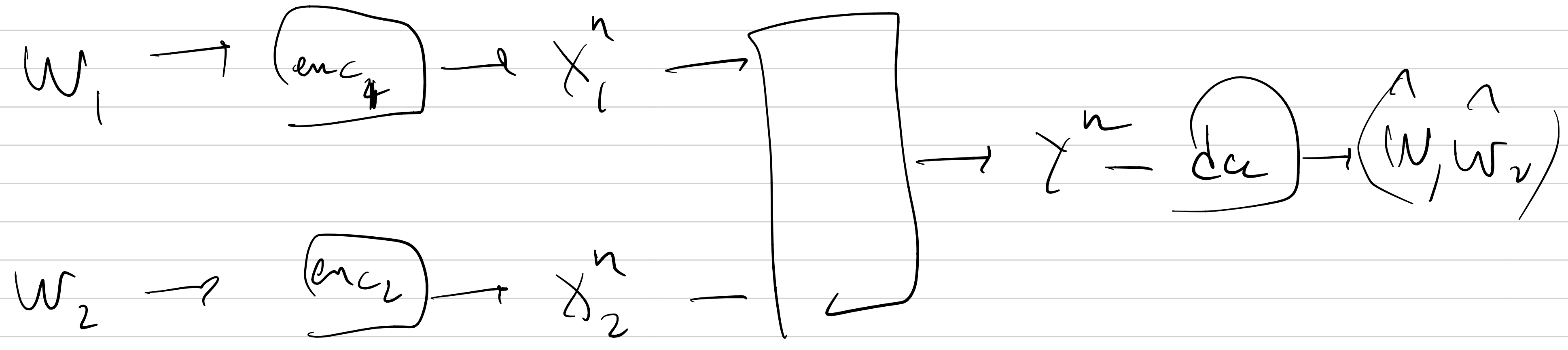
Aside: the setup is different than





"Bad news Thm":

Suppose $(R_1, R_2) \in \mathcal{C}$ $\forall \varepsilon \ \exists$



$W_1 \in \{1 .. M_1\}$ $\quad M_1 \geq 2^{nR_1}$

$W_2 \in \{1 .. M_2\}$ $\quad M_2 \geq 2^{nR_2}$ $\qquad X_1^n, X_2^n$ indep.

↑ uniform & indep

$\Pr\left((\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)\right) < \varepsilon$

$\Rightarrow H(W_1, W_2 | Y^n) < n\varepsilon'$ $\quad$ (Fano).

$$\Rightarrow \left(\begin{aligned} I(W_1; Y^n) &= H(W_1) - H(W_1 | Y^n) \\ &\geq nR_1 - n\varepsilon' = n(R_1 - \varepsilon') \end{aligned} \right)$$

$$I(W_2; Y^n) \geq n(R_2 - \varepsilon')$$

$$\boxed{I(W_1 W_2; Y^n) \geq n(R_1 + R_2 - \varepsilon')}$$

$$n(R_1+R_2-\varepsilon') \le I(W_1W_2;Y^n)$$

$$\le I(X_1^nX_2^n;Y^n)$$

$$= H(Y^n) - H(Y^n|X_1^nX_2^n)$$

$$= H(Y^n) - \sum_{i=1}^{n} H(Y_i|X_{1i}X_{2i}) \Leftarrow$$ Memoryless

$$\le \sum H(Y_i) - \quad ''$$

$$= \sum_{i=1}^{n} I(X_{1i}X_{2i};Y_i)$$

$$n(R_1-\varepsilon') \le I(W_1;Y^n)$$

$$\le I(X_1^n;Y^n)$$

?

$$\cancel{= H(Y^n) - H(Y^n|X_1^n)}$$

$$p(y^n|x_1^nx_2^n) = \prod p(y_i|x_{1i}x_{2i}) \checkmark$$

$$p(y^n|x_1^n) = \prod p(y_i|x_{1i}) \times$$

$$\le I(X_1^n;Y^nX_2^n)$$

$$= \cancelto{0}{I(X_1^n;X_2^n)} + I(X_1^n;Y^n|X_2^n)$$

$$= H(Y^n|X_2^n) - H(Y^n|X_1^nX_2^n)$$

$$= H(Y^n | X_2^n) - \sum H(Y_i | X_{1i} X_{2i})$$

$$\leq \sum H(Y_{1i} | X_2^n) - \text{''}$$

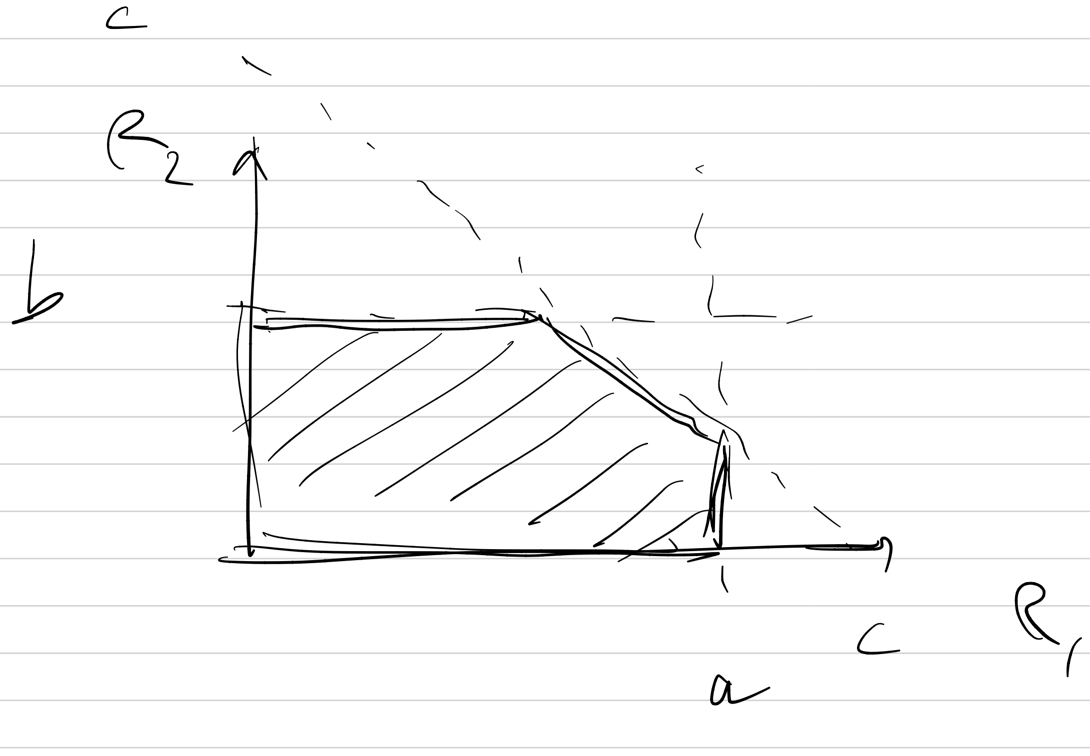
$$\leq \sum H(Y_{1i} | X_{2i}) - \text{''}$$

$$= \sum_i I(X_{1i} ; Y_i | X_{2i})$$

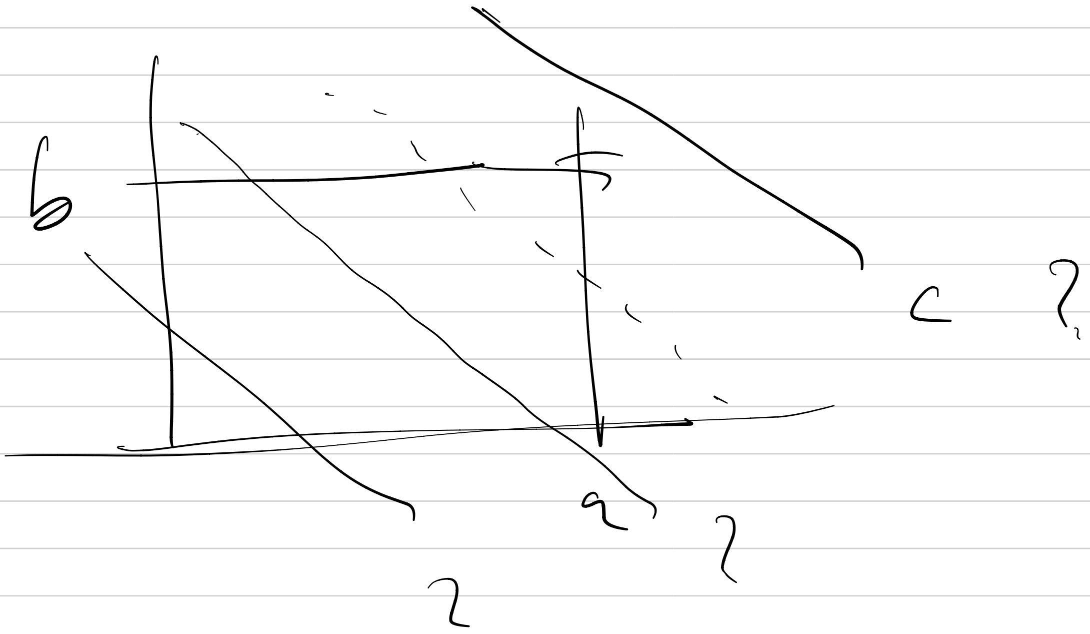
$$= \sum_i I(X_{1i} ; Y_i X_{2i}) \Leftarrow \text{indep of } X_{1i} \,\&\, X_{2i}$$

So far:

$$\begin{cases} n(R_1 + R_2 - \varepsilon') \leq \frac{1}{n} \sum_i I(X_{1i} X_{2i} ; Y_i) = c \\ n(R_1 - \varepsilon') \leq \frac{1}{n} \sum_i I(X_{1i} ; Y_i X_{2i}) = a \\ n(R_2 - \varepsilon') \leq \frac{1}{n} \sum_i I(X_{2i} ; Y_i X_{1i}) = b. \end{cases}$$



Is the diagram faithful?



$\equiv$ is it true that $\max(a,b) \leq c \leq a+b$

Yes: easy to show that if

X_1, X_2 independent, then

$I(X_1;Y X_2) \leq I(X_1 X_2;Y)$ $(a \leq c)$

$I(X_2;Y X_1) \leq$ " $(b \leq c)$

$I(X_1;Y X_2) + I(X_2;Y X_1) \leq I(X_1 X_2;Y)$.

$(a+b \leq c)$

So we have shown that

$(R_1, R_2) \in C$ then there exists

X_1^n, X_2^n independent s.t.

$R_1 + R_2 \leq c$ $\qquad c = \frac{1}{n} \sum \overbrace{I(X_{1i} X_{2i} ; Y_i)}^{c_i}$

$R_1 \leq a$ $\qquad a = \frac{1}{n} \sum I(X_{1i} ; Y_i X_{2i})$ a_i

$R_2 \leq b$ $\qquad b = \frac{1}{n} \sum_i b_i$

let $\mathcal{R}(p_{X_1} p_{X_2}) = \{ (R_1, R_2) :$

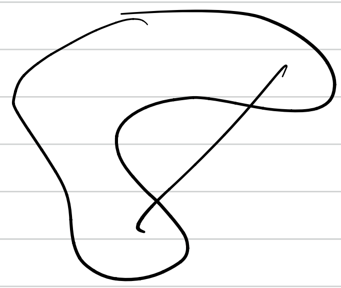
$$R_1 \leq I(X_1 ; Y X_2)$$
$$R_2 \leq I(X_2 ; Y X_1)$$
$$R_1 + R_2 \leq I(X_1 X_2 ; Y) \}.$$

pentagonal region.

let $$\mathcal{R} = \text{convex Hull}\left(\bigcup_{p_{X_1}, p_{X_2}} \mathcal{R}(p_{X_1} p_{X_2}) \right)$$

<u>Thm</u> :- $\underline{C \subseteq \mathcal{R}}$. (Bad news).

<u>Aside</u> on convex sets, convex Hulls.



if A, B are convex sets
$A \cap B$ is also convex.

$\Rightarrow$ if $\{A_\alpha\}$ a collection of convex sets

$\Rightarrow$ $\bigcap_\alpha A_\alpha$ is also convex.

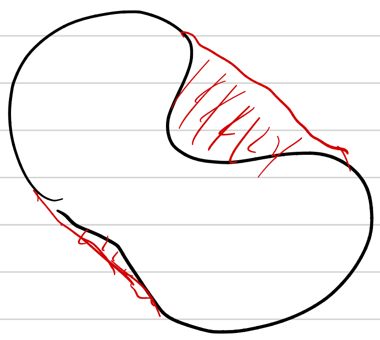
given a set S consider <u>all convex sets which</u> <u>include S</u>, let S_0 = intersection of all these the intersection S_0 is convex and it includes S. As if any convex set A includes S then $A \subset S_0$ to summarize:

$\{$ ① S_0 is convex & $S \subseteq S_0$

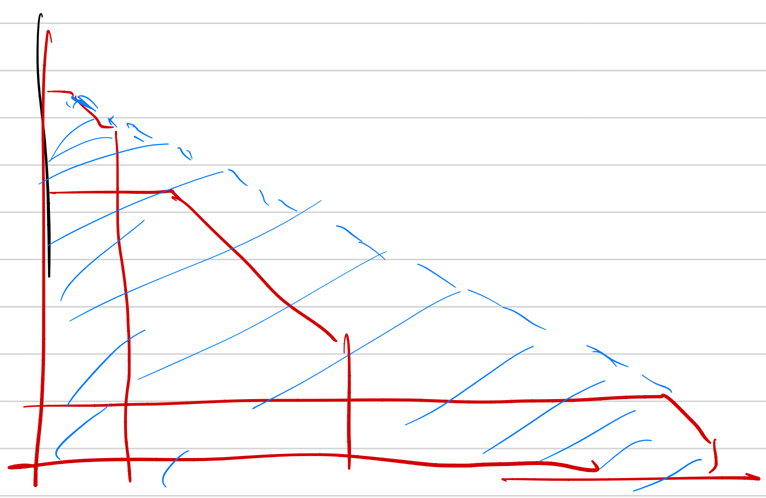
② if A is convex & $S \subset A \Rightarrow A \subseteq S_0$.

$\Leftrightarrow$ S_0 is the smallest convex set which includes S.

$=$ convex-hull of S.

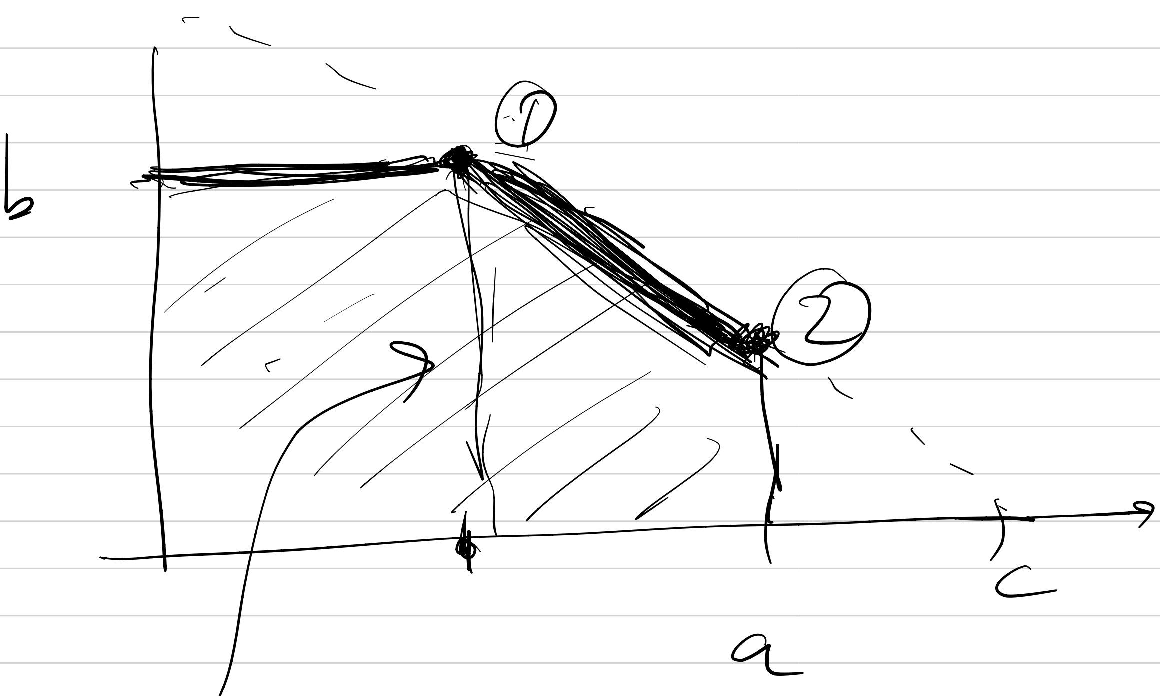


R is a union of convex pentagons.



Pf: we have seen that $(R_1, R_2) \in C$

$\Rightarrow$



(R_1, R_2)

$a = \frac{1}{n} \sum a_i$

$b = \frac{1}{n} \sum b_i$

$c = \frac{1}{n} \sum c_i$

it is sufficient to show

that $1 \in C.H.(\cup R(p_{X_1} p_{X_2}))$ $\Leftarrow$

$2 \in$ "

what are the coordinates of 1 :

$R_1 = c - b, \quad R_2 = b$

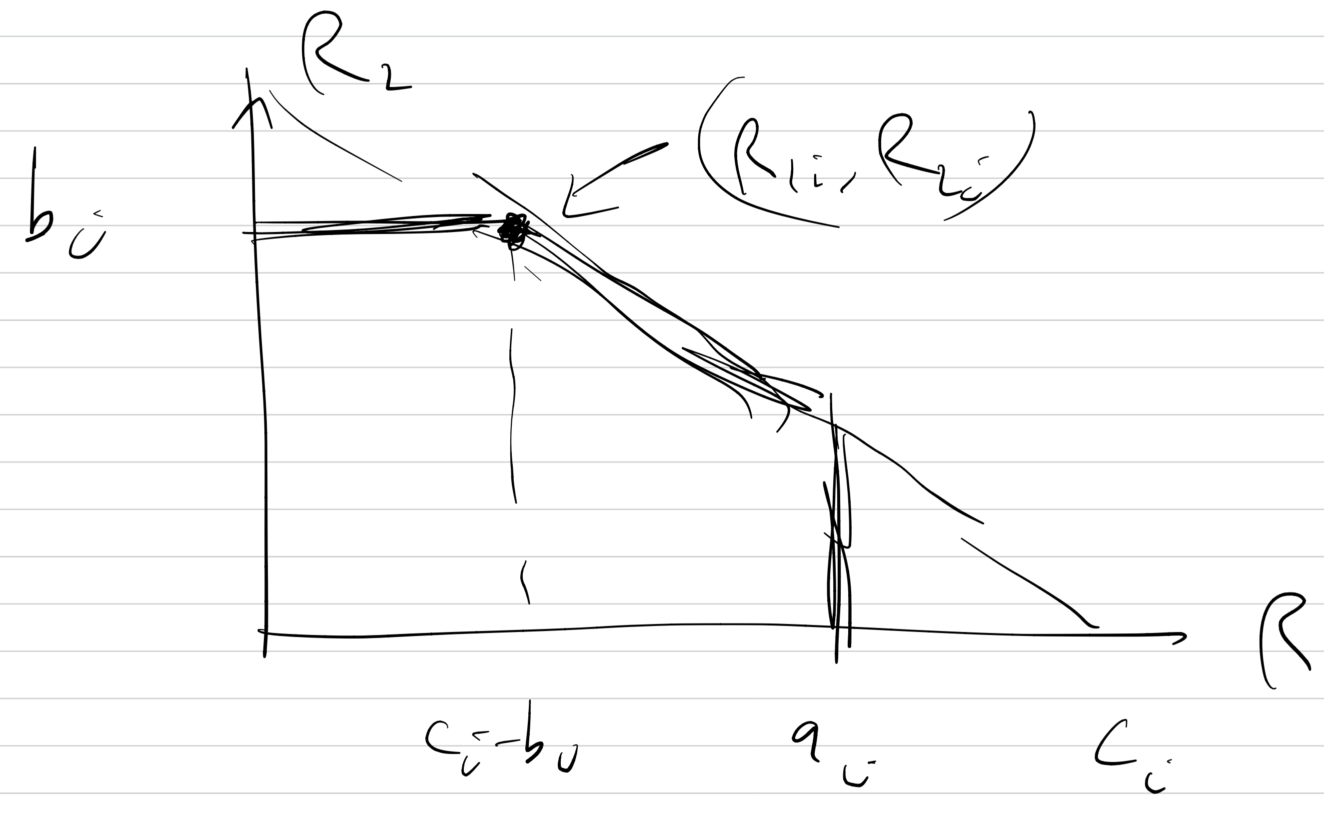
$$R_1 = \frac{1}{n} \sum_{i=1}^{n} \underbrace{I(X_{1i} X_{2i} ; Y) - I(X_{2i} ; Y_i | X_{1i})}_{R_{1i}}$$

$$R_2 = \frac{1}{n} \sum_{i=1}^{n} \underbrace{I(X_{2i} ; Y_i | X_{1i})}_{R_{2i}}$$

$$\underline{(R_1, R_2)} = \frac{1}{n} \sum_{i=1}^{n} \underline{(R_{1i}, R_{2i})}$$

So if we show $(R_{1i}, R_{2i}) \in C.H.(\quad)$

but $(R_{1i}, R_{2i}) \in \mathcal{R}(P_{X_{1i}} P_{X_{2i}})$



Note that the computation of

$$CH\left(\bigcup_{P_{X_1}P_{X_2}} \mathcal{R}(P_{X_1}P_{X_2})\right) \text{ is "single-letter"}$$

we also have, (surprise):

<u>Thm</u>: (Good news):

$$C \supseteq C.H.\left(\bigcup_{P_{X_1}P_{X_2}} \mathcal{R}(P_{X_1}P_{X_2})\right)$$

<u>Corollary</u>: $C = C.H.\left(\bigcup \mathcal{R}(P_{X_1}P_{X_2})\right)$.

Pf: random coding: fix p_{X_1}, p_{X_2}.

generate 2^{nR_1} sequences independents

$$\{X_1^n(m_1) : m_1 = 1 \dots 2^{nR_1}\}$$

iid $\sim p_{X_1}$

generate 2^{nR_2} sequences independents

$$\{X_2^n(m_2) : m_2 = 1 \dots 2^{nR_2}\}$$

iid $\sim p_{X_2}$

enc_1 will map $m_1 \to X_1^n(m_1)$

enc_2 " " $m_2 \to X_2^n(m_2)$

what about the decoder?

$\mathrm{dec}(y^n)$: search for $\hat{m}_1, \hat{m}_2$ s.t.

$$(X_1^n(m_1), X_2^n(m_2), y^n) \in T(\underbrace{p_{X_1X_2Y}}_{p_{X_1} p_{X_2} p_{Y|X_1X_2}}, n, \delta).$$

if we find only one such $(\hat{m}_1, \hat{m}_2)$ declare it.

otherwise declare randomly.

Probability of error: suppose $m_1 = 1, m_2 = 1$

$$\leq \Pr\left((X_1^n(1), X_2^n(1), Y^n) \notin T \right) \quad (1)$$

$$+ \sum_{m_2=2}^{M_2} \Pr\left((X_1^n(1), X_2(m_2), Y^n) \in T \right) \quad (2)$$

$$+ \sum_{m_1=2}^{M_1} \Pr\left(X_1^n(m_1), X_2^n(1), Y^n) \in T \right) \quad (3)$$

$$+ \sum_{m_1=2}^{M_1} \sum_{m_2=2}^{M_2} \Pr\left((X_1^n(m_1) X_2^n(m_2), Y^n) \in T \right) \quad (4)$$

$\Pr(1) \searrow 0$ because $(X_1^n(1), X_2^n(1), Y^n)$ iid $p_{X_1} p_{X_2} p_{Y|X_1X_2}$

$\Pr(2)$: 2^{nR_2} elements in the sum each has

$$\Pr\left((X_1^n(1), X_2^n(m_2), Y^n) \in T(p_{X_1} p_{X_2} p_{Y|X_1X_2}) \right)$$

iid $p_{X_1 Y}\, p_{X_2} = q_{X_1 X_2 Y}$

$$\approx 2^{-n(D(p\|q) \pm \delta..)}$$

$D(p\|q) = D(p_{X_1X_2Y} \| p_{X_1Y}\, p_{X_2})$

$= I(X_2; Y X_1)$

so in (2) we have

$\underbrace{2^{nR_2}}$ terms each of which is $\underbrace{2^{-n\underbrace{I(X_2;YX_1)}}}$

so (2) $\searrow 0$ if $R_2 < I(X_2; YX_1)$.

Similarly (3) $\searrow 0$ if $R_1 < I(X_1; YX_2)$

(4) $\searrow 0$ $R_1 + R_2 < I(X_1X_2; Y)$

So if $(R_1, R_2) \in \underbrace{\mathcal{R}(p_{X_1}\, p_{X_2})}$ then

the $p_e \searrow 0$.

$\Rightarrow \mathcal{R}(p_{X_1}\, p_{X_2}) \subseteq \mathcal{C} \quad \forall\, p_{X_1}\, p_{X_2}$

$\Rightarrow \bigcup_{p_{X_1} p_{X_2}} \mathcal{R}(\,''\,) \subseteq \mathcal{C}$

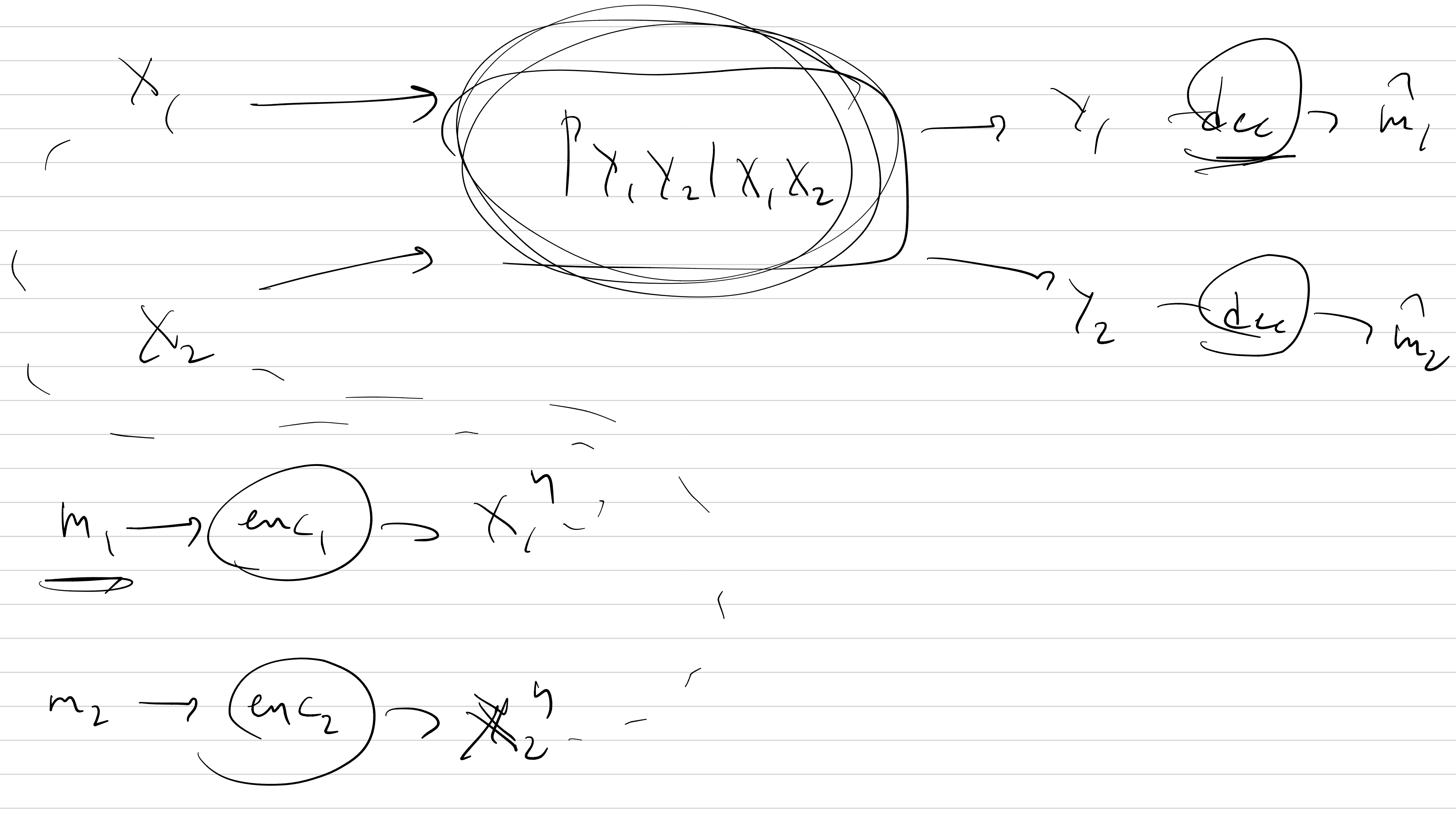
Since $\mathcal{C}$ is convex we see that

$C.H.\left(\bigcup \mathcal{R}(\quad) \right) \subseteq \mathcal{C}$. //

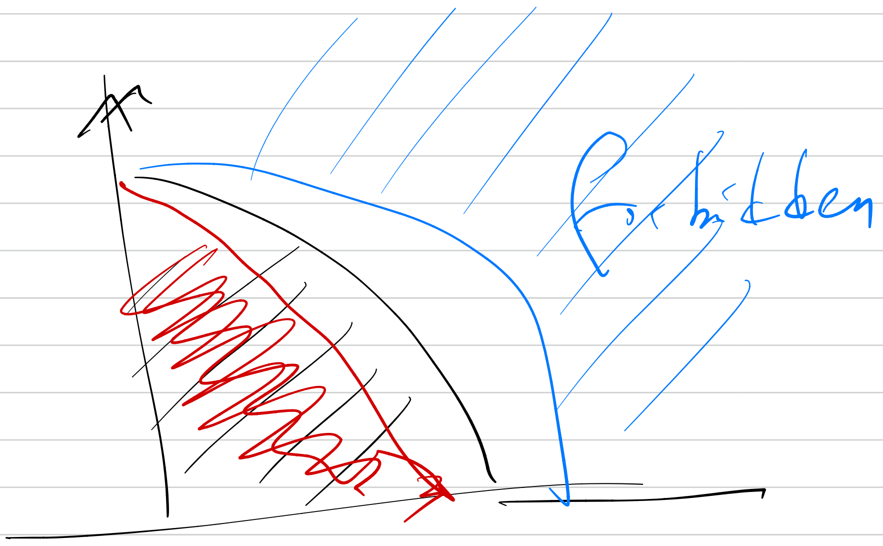
the impression so far:

whenever we are given a communication scenario, the fundamental limits on the rates can be found exactly by computing some mutual informations and/or entropies

The impression is wrong: there are natural setups where we do not know the fundamental limits



Q: what are (R_1, R_2) pairs that may result in reliable comm.?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