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ANALYSIS OF CRACK FORMATION AND CRACK GROWTH IN CONCRETE BY MEANS OF FRACTURE MECHANICS AND FINITE ELEMENTS

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ABSTRACT

A method is presented in which fracture mechanics is introduced into finite element analysis by means of a model where stresses are assumed to act across a crack as long as it is narrowly opened. This assumption may be regarded as a way of expressing the energy absorption G_c in the energy balance approach, but it is also in agreement with results of tension tests. As a demonstration the method has been applied to the bending of an unreinforced beam, which has led to an explanation of the difference between bending strength and tensile strength, and of the variation in bending strength with beam depth.

Une mêthode est présentêe, par laquelle la méchanique des ruptures est introduite dans l'analyse des éléments finis à l'aide d'un modêle, oũ les contraintes sont supposées d'opérer sur les côtés d'une fissure tant que cette fissure est ètroite.

Cette hypothêse peut être considérée comme un moyen d'exprimer l'absorption G_c d'ênergie en usant l'approche de l'équilibre d'énergie. Cette hypothèse est aussi justifiée par les résultats des essais de tension.

Pour en prouver la validité, cette méthode a été appliquée au fléchissement d'une poutre non armée et fournit une explication de la différence entre la résistance au moment de flexion et la rêsistance à l'effort de tension, ainsi que de la variation de la résistance au moment de flexion en fonction de la profondeur de la poutre.

Importance of cracks and crack growth

Crack formation and crack growth play an important part in the performance of unreinforced and reinforced concrete. Examples of this are

crack spacing and crack width in bending

shear chracks and their effect on shear capacity

cracking moment of reinforced and unreinforced beams

microcracks in compression and compression failure.

A rational design in these cases ought to be based on realistic theoretical models, which take crack formation and crack propagation into account. So far no such models have been available. Consequently the design methods have had to be based on empirical research, supported by simplified models.

Recent advances within fracture mechanics and finite element methods (FEM) have now given us a possibility of analysing crack growth. Fracture mechanics gives the fundamental rules for crack propagation and FEM makes it possible to apply these rules to complicated cases.

The cases we wish to analyse are rather complicated, as they involve diverse phenomena, such as

formation and propagation of cracks

two or more parallel cracks

bent shear cracks

shrinkage strains

interaction between concrete and reinforcement

interaction between cement matrix and aggregate.

It is therefore necessary to use FEM and also to try and find a method which simplifies the analysis as much as possible.

Proposed approach

There are many methods to choose from fracture mechanics, e.g.

the stress intensity factor approach

the energy balance approach

the "strip-yield" model according to Dugdale

the cohesive force model according to Barenblatt.

The different methods are known to give coherent results.

In the stress intensity factor approach the stresses near the crack tip are studied. These stresses theoretically approach infinity at the crack tip according to the expression $\sigma = K/\sqrt{2\pi r}$, where r is the distance from the crack tip and K is a coefficient, the stress intensity factor, depending on the load, the crack dimensions, etc. When K reaches a critical value K_c, the crack propagates.

The stress intensity factor approach has been used a great deal in FEM analysis. The direct method requires a FEM mesh with very small elements close to the crack tip, which limits its applicability to complicated problems. Indirect and special methods permit the use of greater elements. The methods cannot explain the formation of cracks, only the propagation.

In the energy balance approach it is assumed that a certain amount of energy G_c is absorbed by the formation of a unit area of crack surface. When a crack propagates a certain amount of stored energy is released. The crack propagates when the released energy is equal to or greater than the absorbed energy. FEM has been used to determine the energy release rate in the energy balance approach, see e.g. /5/. This enables the use of a FEM mesh with rather large elements. The formation of cracks cannot be explained.

In the Dugdale model it is assumed that there is a plastic zone near the crack tip according to Fig. 1. Within the plastic zone a stress equal to the yield strength σ_Y acts across the crack. The Barenblatt model is similar to the Dugdale model, but the stress is assumed to vary with the deformation. It does not seem to have been used in finite element analysis.

The basic idea of the model we propose is demonstrated in Fig. 2. It is in some respects similar to the Barenblatt model. The model is described only for mode I (the opening mode), but it may also be applied to modes II and III.

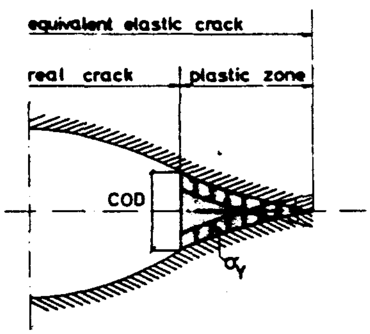


Fig. 1 The Dugdale model for crack tip plasticity

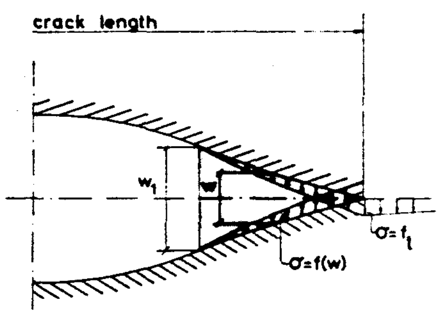


Fig. 2 Proposed model

The crack is assumed to propagate when the stress at the crack tip reaches the tensile strength f_t. When the crack opens the stress is not assumed to fall to zero at once, but to decrease with increasing crack width w, for example according to Fig. 3. At the crack width w_1 the stress has fallen to zero. For that part of the crack where $w < w_1$, the "crack" in reality corresponds to a microcraced zone with some remaining ligaments for stress transfer. As there is a stress to be overcome in opening the crack, energy is absorbed. The amount of energy absorbed per unit crack area in widening the crack from zero to or beyond w_1 is

$$\int_0^{w_1} \sigma dw$$

and corresponds to the area between the curve and the coordinate axis' in Fig. 3.

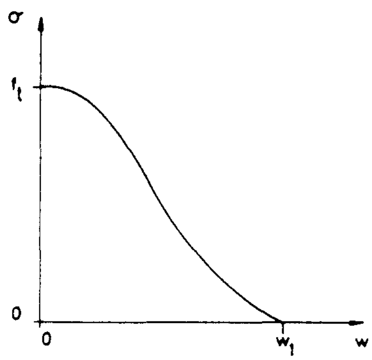


Fig. 3 Assumed variation of stress σ with crack width w, general case

We now choose the curve in Fig. 3 so that

$$\int_0^{w_1} \sigma dw = G_c \tag{1}$$

which means that the energy absorbed per newformed unit crack area is the same as in the energy balance approach. The model of Fig. 2 may thus be looked upon as a way of expressing the energy balance approach.

At the same time the assumption of Fig. 2 may be looked upon as a reality. Stresses may be present in a microcracked zone as long as the corresponding displacement is small. This has been clearly demonstrated in tension tests, using a very rigid testing equipment, e.g., by Evans and Marathe /4/; cf. Fig. 5.

By the application of the proposed model the curve σ(w) may be chosen in different ways, e.g. according to Figs. 4a, b or c, which all show simple mathematical relations. For typical yielding materials, like mild steel, Fig. 4a seems to be the best choice. It corresponds exactly to the Dugdale model with $f_t = \sigma_Y$ and w_1 = COD at initiation of crack growth. The discontinuity may give rise to some problems by the application in FEM, but they are not serious.

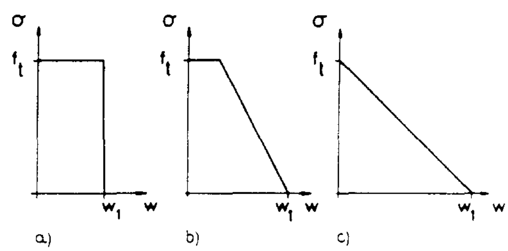


Fig. 4 Examples of possible assumptions of variation of stress σ with crack width w in practical applications

For concrete it seems that Fig. 4c is the best choice as it corresponds reasonably well with tension test results /4/, cf. Fig. 5. It is also simple, continuous and suitable for FEM analysis. For our purpose we have therefore chosen Fig. 4c.

We then obtain

$$\int_0^{w_1} \sigma dw = f_t w_1/2$$

or from (1),

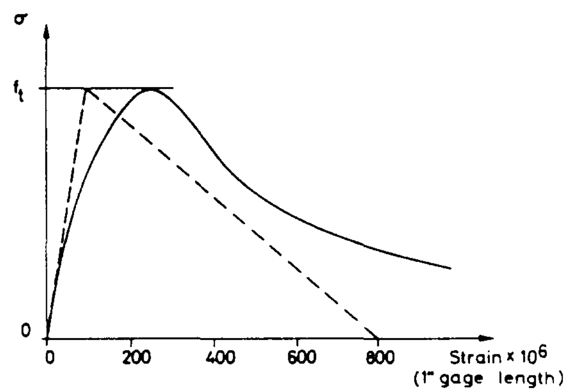
$$w_1 = 2G_c/f_t \tag{2}$$

For ordinary concrete G_c/f_t seems to be of the order 0.005 -

0.01 mm, cf. /1/, and thus w_1 of the order 0.01 - 0.02 mm. In the application we further assume that the concrete is linear-elastic until f_t is reached.

Fig. 5 shows a comparison between our assumptions with G_c/f_t = 0.01 mm, E/f_t = 10 000 and a tension test from /4/ with a gage length of 1" (25 mm). This corresponds to a theoretical average elongation over the gage length when σ reaches 0, i.e. $w = w_1$, of $2 \cdot 0.01/25 = 800 \cdot 10^{-6}$. The assumptions seem to agree reasonably with the test result. A lower value of E/f_t would have improved the agreement, but from the point of view of the energy balance approach the E-value corresponding to unloading is most important and this justifies the choice E/f_t = 10 000.

Fig. 5 A test result from a tensile test according to Evans & Marathe /4/, compared to a corresponding assumed relation by the analysis



A special feature of the proposed method is that it explains not only the growth of existing cracks, but also the formation of new cracks, as it is assumed that cracks start forming when the tensile stress reaches f_t, i.e., the same criterion is used for formation and propagation of cracks.

The analysis can be performed with a rather coarse mesh, as in the example below, because there are no stress singularities and the amount of absorbed energy is not very sensitive to the mesh size. The possibility of using a coarse mesh means that rather complicated problems can also be treated without using too many elements.

Application to an unreinforced beam in bending

In order to study the applicability of the method the following case has been analysed /2/.

An unreinforced concrete beam with a constant rectangular cross-section is loaded by a pure bending moment M according to Fig. 6. When the bending moment reaches a value M_0 the tensile stress in the bottom fibre reaches f_t. As we assume that the concrete cannot take higher tensile stresses than f_t, cracks will form and start opening when M is increased above M_0. We will now study how these cracks grow when the bending moment increases. In order to simplify the calculations we assume that only one crack opens, and that this happens at the section of symmetry.

The finite element mesh used for the calculation is shown in Fig. 6. The bending moment M is applied as a couple of forces at the

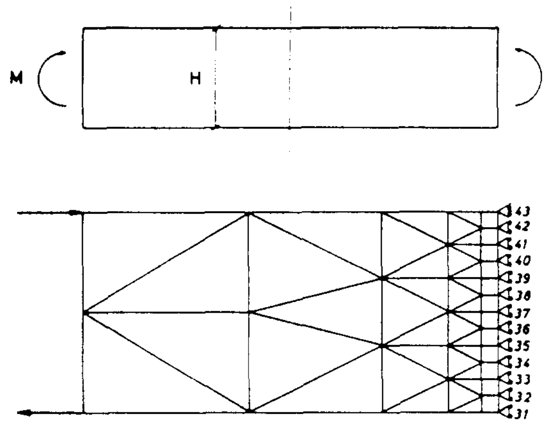


Fig. 6 Bent rectangular beam and a corresponding FEM representation

left end of the beam. The crack is assumed to open at the section to the right, which is the section of symmetry.

M_0 is the moment which gives $\sigma_{31} = f_t$, where σ_{31} is the stress at point 31. M_0 would be the failure moment if the material were elastic and perfectly brittle. When M is raised above M_0 the crack starts opening at point 31. At that point we introduce a force corresponding to the relation between stress σ and crack width w according to Fig. 4c. With this new finite element system we can calculate the stress at point 32 and we can determine that value $M = M_1$, which gives a stress $\sigma_{32} = f_t$. We can now introduce another force at point 32 and calculate a moment $M = M_2$, giving $\sigma_{33} = f_t$ etc. By proceeding in the same way we get a relation between crack depth and applied moment according to Fig. 7.

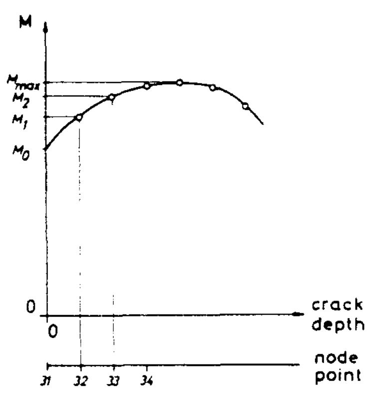


Fig. 7 Calculated bending moment M versus crack depth

When the crack grows the corresponding bending moment reaches a maximum value M_{max} whereupon it starts decreasing. As the maximum value is reached the structure becomes unstable if M is kept constant, and it fails suddenly as the crack propagates.

The relation M_{max}/M_0 is the same as the relation between bending strength and tensile strength, as M_0 is the moment which makes the maximum bending stress in the uncracked section equal to the tensile strength.

It can be shown that the behaviour of the beam depends on the parameter H/l_c, where H is the beam depth and l_c is a critical length, defined by

$$l_c = EG/f_t^2 \tag{3}$$

As the realtion

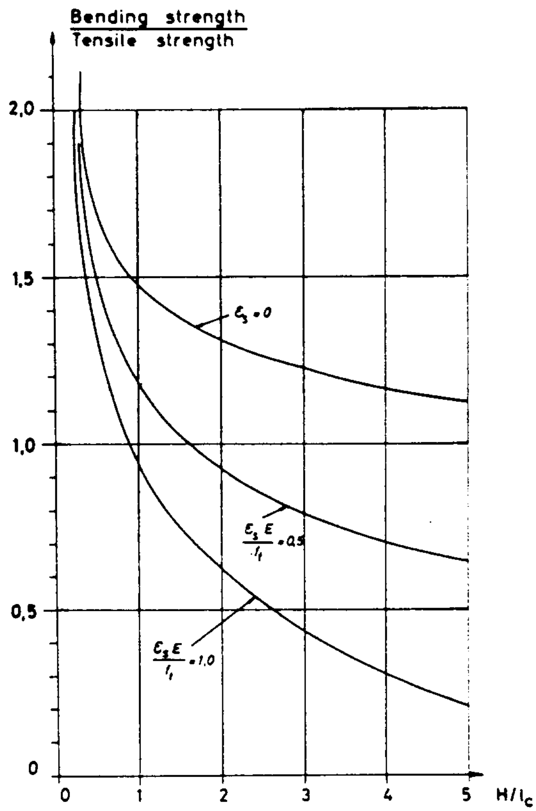
$$EG_c = K_c^2$$

holds for plain stress and approximately for plain strain, we may also write

$$l_c = (K_c/f_t)^2 \tag{4}$$

Fig. 8 shows the results of the above analysis as well as of an analysis where shrinkage strains ε_s according to Fig. 9 have been taken into account.

Fig. 8 Theoretical variation of ratio between bending and tensile strength with beam depth H and $l_c = (K_c/f_t)^2 = EG_c/f_t^2$



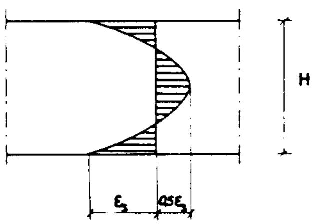


Fig. 9 Assumed distribution of shrinkage strains

It must be remembered that the results in Fig. 8 correspond to a simple FEM model where only one crack is assumed to open, independent of the stresses in the other parts of the beam. A more realistic model with cracks opening in all places where f_t is exceeded will give somewhat different results with higher values of M_{max}/M_o, especially where shrinkage strains are present.

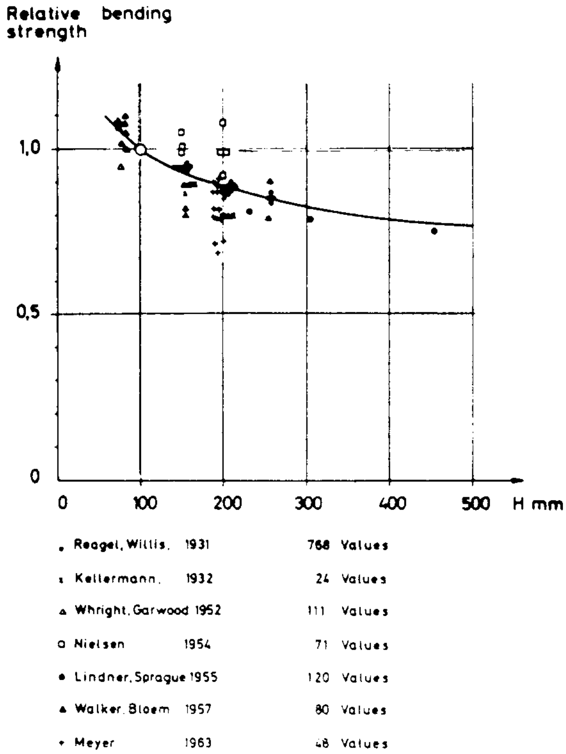


Fig. 10 Test results of bending strength versus beam depth, summarized by Meyer /3/, compared to theoretical curve for l_c = 100 mm

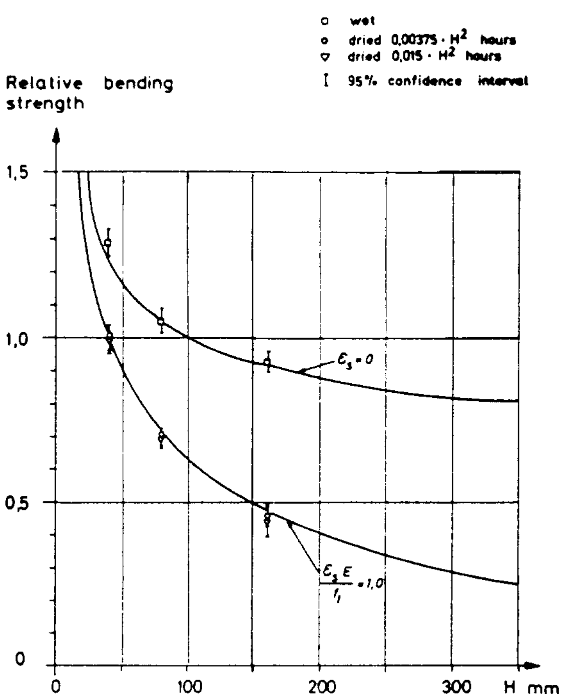


Fig. 11 Test results of bending strength versus beam depth for one quality of concrete, tested wet and dried in 45% RH, compared to theoretical curves for l_c = 100 mm

Fig. 10 shows a comparison between theoretical values according to Fig. 8 and test results summarized by Mayer /3/. The theoretical curve is shown for $l_c = EG_c/f_t^2 = 10\ 000 \cdot 0.01 = 100$ mm, corresponding to the values used in Fig. 5. It has been assumed that there is no shrinkage.

Fig. 11 shows a comparison between theoretical values according to Fig. 8 and our own test results. Regarding the influence of shrinkage it must be noticed that the test specimen had a square cross-section, drying in all directions, whereas the theoretical curve is valid for a specimen drying only upwards and downwards, and that creep was not taken into account in the calculations.

In spite of its simplification, the model seems to be able to explain the test results.

Conclusion

The proposed method of combining fracture mechanics and finite element analysis seems to yield realistic results regarding crack formation and propagation as well as regarding failure even if a coarse element mesh is used. This opens up the possibility of studying complicated problems with a limited amount of computer work.

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