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Exercise #9: Eigenvalue analysis

Problem 1

For the two-storey shear building shown in Figure 1.1, determine the following:

1. The mass and stiffness matrices of the building.
2. Determine the natural vibration frequencies and modes. Express the frequencies in terms of m, EI, and h.
3. Verify that the modes satisfy the orthogonality properties.
4. Normalize each mode so that the roof displacement is unity. Sketch the modes and identify the associated natural frequencies.
5. Normalize each mode so that the modal mass Mn has unit value. Compare these modes with those obtained in part (4) and comment on the differences.

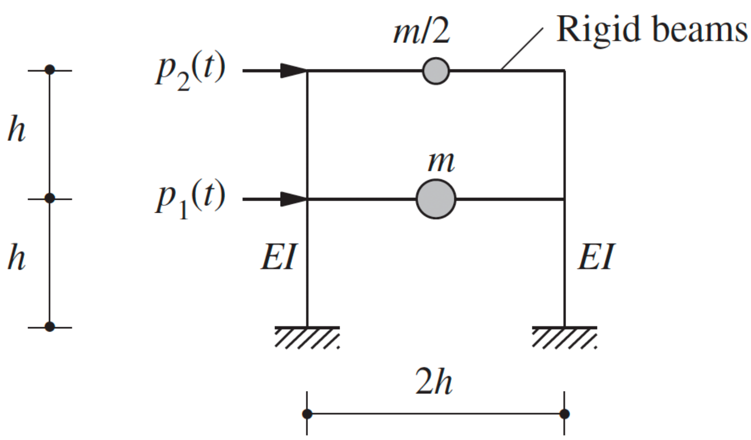


Figure 1.1

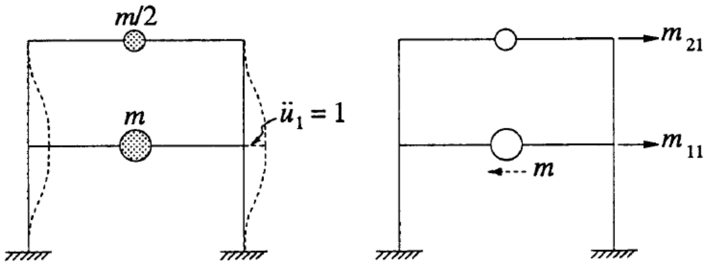
Solution

Question 1:

Determine the mass matrix:

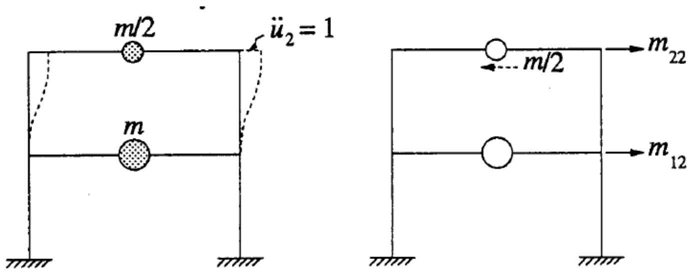
Apply $\ddot{u}_1 = 1, \ddot{u}_2 = 0$

Therefore, from the figure below, $m_{11} = m$ and $m_{21} = 0$



Apply $\ddot{u}_2 = 1, \ddot{u}_1 = 0$

Therefore, from the figure below, $m_{22} = \frac{m}{2}$ and $m_{12} = 0$



Therefore, the mass matrix is as follows:

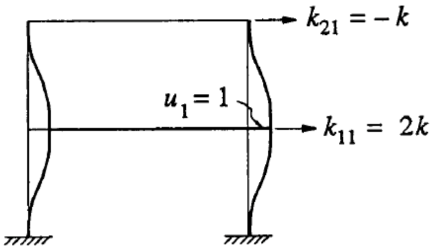
$$\mathbf{m} = m\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Concerning the stiffness matrix:

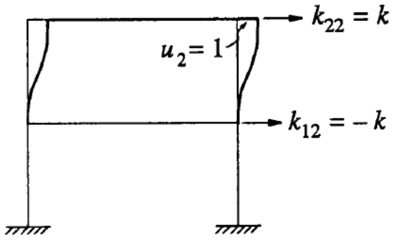
The storey stiffness is as follows:

$$k = 2\left(\frac{12EI}{h^3}\right) = \frac{24EI}{h^3}$$

Apply $u_1 = 1, u_2 = 0$ and determine k_{i1}



Apply $u_2 = 1, u_1 = 0$ and determine k_{i2}



Therefore, the stiffness matrix is as follows:

$$k = k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \frac{24EI}{h^3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Question 2:

Determine the natural frequencies and modes:

$$\det[\mathbf{k} - \omega_n^2\mathbf{m}] = 0 \Rightarrow$$

$$\det\left(k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega_n^2 m\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}\right) = 0 \Rightarrow$$

$$\det\left(\begin{bmatrix} 2k - \omega_n^2 m & -k \\ -k & k - 0.5\omega_n^2 m \end{bmatrix}\right) = 0 \Rightarrow$$

$$2k^2 - 2km\omega_n^2 + 0.5m^2\omega_n^4 - k^2 = 0 \Rightarrow$$

$$k^2 - 2km\omega_n^2 + 0.5m^2\omega_n^4 = 0 \Rightarrow$$

$$\omega_n^2 = \frac{k}{m}\left(2 \pm \sqrt{2}\right) \Rightarrow$$

$$\omega_1 = 0.765\sqrt{\frac{k}{m}}, \quad \omega_2 = 1.848\sqrt{\frac{k}{m}}$$

First mode:

$$[\mathbf{k} - \omega_1^2\mathbf{m}]\phi_1 = \mathbf{0}$$

$$k\begin{bmatrix}\sqrt{2} & -1\\ -1 & 1/\sqrt{2}\end{bmatrix}\begin{Bmatrix}\phi_{11}\\ \phi_{21}\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Select, $\phi_{11}=1 \Longrightarrow \phi_{21}=\sqrt{2}$

$$\phi_1=\begin{Bmatrix}1\\ \sqrt{2}\end{Bmatrix}$$

Second Mode:

$$[\mathbf{k}-\omega_2^2\mathbf{m}]\phi_2=\mathbf{0}$$

$$k\begin{bmatrix}-\sqrt{2} & -1\\ -1 & -1/\sqrt{2}\end{bmatrix}\begin{Bmatrix}\phi_{12}\\ \phi_{22}\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Select, $\phi_{12}=1 \Longrightarrow \phi_{21}=-\sqrt{2}$

$$\phi_2=\begin{Bmatrix}1\\ -\sqrt{2}\end{Bmatrix}$$

Question 3:

Verify orthogonality of modes:

$$\phi_1^T\mathbf{m}\phi_2=\langle 1 \quad \sqrt{2}\rangle\cdot m\cdot\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}\cdot\begin{Bmatrix}1\\ -\sqrt{2}\end{Bmatrix}=m\langle 1 \quad \sqrt{2}\rangle\cdot\begin{Bmatrix}1\\ -\sqrt{2}\end{Bmatrix}=0$$

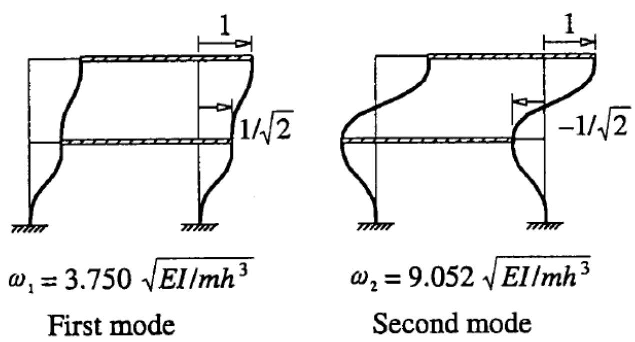
$$\phi_1^T\mathbf{k}\phi_2=\langle 1 \quad \sqrt{2}\rangle\cdot k\cdot\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\cdot\begin{Bmatrix}1\\ -\sqrt{2}\end{Bmatrix}=k\langle 1 \quad \sqrt{2}\rangle\cdot\begin{Bmatrix}2+\sqrt{2}\\ -1-\sqrt{2}\end{Bmatrix}=0$$

Question 4:

Normalize modes to unit value at roof

$\phi_1=\begin{Bmatrix}1/\sqrt{2}\\ 1\end{Bmatrix}$ and $\phi_2=\begin{Bmatrix}-1/\sqrt{2}\\ 1\end{Bmatrix}$

Therefore,



Question 5:

Normalize modes so that $M_n = 1$.

$$M_1 = \phi_1^T \mathbf{m} \phi_1 = \langle 1 \quad \sqrt{2} \rangle \cdot m \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix} = 2m$$

$$M_2 = \phi_2^T \mathbf{m} \phi_2 = \langle 1 \quad -\sqrt{2} \rangle \cdot m \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix} = 2m$$

Divide ϕ_1 from Question 2 by $\sqrt{2m}$ and ϕ_2 from Question 2 by $\sqrt{2m}$ to obtain the normalized modes:

$$\phi_1 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 1/\sqrt{2} \\ 1 \end{Bmatrix}, \quad \phi_2 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 1/\sqrt{2} \\ -1 \end{Bmatrix}$$

These modes differ from these obtained in Question 4, only by a scale factor; the shapes of the two sets of modes are the same.