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Exercise #2: Response to harmonic and periodic motion

Problem 1

The steady-state acceleration amplitude of a structure caused by an eccentric-mass vibration generator was measured for several excitation frequencies. The data are as follows:

Frequency [Hz]	Acceleration [$10^{-3}g$]	Frequency [Hz]	Acceleration [$10^{-3}g$]
1.337	0.68	1.500	7.10
1.378	0.90	1.513	5.40
1.400	1.15	1.520	4.70
1.417	1.50	1.530	3.80
1.438	2.20	1.540	3.40
1.453	3.05	1.550	3.10
1.462	4.00	1.567	2.60
1.477	7.00	1.605	1.95
1.487	8.60	1.628	1.70
1.493	8.15	1.658	1.50
1.497	7.60	-	-

Determine the natural frequency and damping ratio of the structure.

Problem 2

An SDF system is excited by a sinusoidal force. At resonance, the amplitude of displacement was measured to be $50.8mm$. at an exciting frequency of one-tenth ($1/10$) the natural frequency of the system, the displacement amplitude was measured to be $5.08mm$. Estimate the damping ratio of the system.

Problem 3

In a forced vibration test under harmonic excitation, it was noted that the amplitude of motion at resonance was exactly four times the amplitude at an excitation frequency 20% higher than the resonant frequency. Determine the damping ratio of the system.

Problem 4

An SDF undamped system with a period T_n is subjected to a force $p(t)$ consisting of a sequence of two impulses, each of magnitude I , as shown in the figure below.

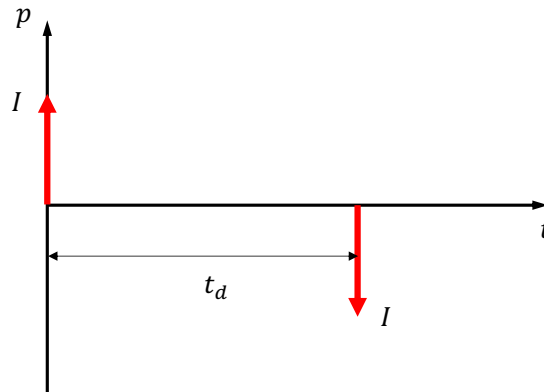
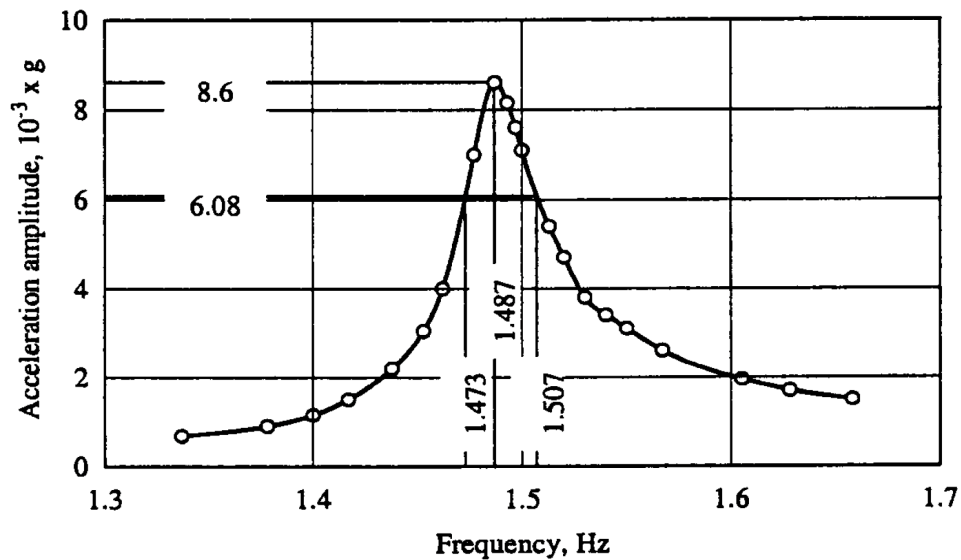


Figure 4.1. external force

1. Plot the displacement response of the system for $\frac{t_d}{T_n} = \frac{1}{8}, \frac{1}{4}$ and 1. For each case show the response to individual impulses and the combined response.
2. Plot $u_o \div (I/m\omega_n)$ as a function of t_d/T_n . Indicate separately the maximum occurring at $t \leq t_d$ and $t \geq t_d$. Such a plot is called the response spectrum for this excitation.

Problem 1 - Solution

The given data is plotted in the form of the frequency response curve as shown in the figure below:



a) Natural frequency

The frequency response curve peaks at $f_n = 1.487\text{Hz}$. Assuming a small damping ratio, this value is the natural frequency of the SDF system.

b) Damping ratio

The acceleration at the peak is $r_{peak} = 8.6 \times 10^{-3}\text{g}$. Draw a horizontal line at $r_{peak} \div \sqrt{2} = 6.08 \times 10^{-3}\text{g}$ to obtain f_a and f_b (in Hz):

$$f_a = 1.473 \text{ Hz and } f_b = 1.507\text{Hz}$$

Then,

$$\zeta = \frac{f_b - f_a}{2f_n} = \frac{1.507 - 1.473}{2(1.487)} = 0.0114 = 1.14\%$$

Problem 2 - Solution

At $\omega = \omega_n$, we get resonance. As such,

$$u_o = \frac{(u_{st})_o}{2\zeta} = 50.8mm \quad (2.1)$$

At $\omega = 0.1\omega_n$, the following equation holds true,

$$u_o \cong (u_{st})_o = \frac{p_o}{k} = 5.08mm \quad (2.2)$$

Substituting $(u_{st})_o = 5.08mm$ in Eq. (2.1) and solving it with respect to the damping ratio,
 $\zeta = 0.05 = 5\%$

Problem 3 - Solution

Assuming that damping is small enough to justify the approximation that the resonant frequency is ω_n and the resonant amplitude of R_d is $1/2\zeta$, then the given data implies the following:

$$(u_o)_{\omega=\omega_n} = (u_{st})_o \frac{1}{2\zeta} \quad (3.1)$$

$$(u_o)_{\omega=1.2\omega_n} = (u_{st})_o \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(1.2)]^2}} \quad (3.2)$$

Hence,

$$(u_o)_{\omega=1.2\omega_n} = (u_{st})_o \frac{1}{\sqrt{[1-(1.2)^2]^2 + [2\zeta(1.2)]^2}} \quad (3.3)$$

Combining Equations (3.1) and (3.3),

$$\frac{1}{(2\zeta)^2} \left(\frac{(u_o)_{\omega=1.2\omega_n}}{(u_o)_{\omega=\omega_n}} \right)^2 = \frac{1}{(-0.44)^2 + (2.4\zeta)^2} \quad (3.4)$$

For

$$\frac{(u_o)_{\omega=1.2\omega_n}}{(u_o)_{\omega=\omega_n}} = \frac{1}{4}$$

Equation (3.4) gives,

$$64\zeta^2 = 0.1935 + 5.76\zeta^2 \Rightarrow \zeta = 0.0576 = 5.76\%$$

Observations: Considering the assumption, we made in Equation (3.1) for a small damping ratio is deemed reasonable. Otherwise, we would have to use the exact resonant frequency, which is equal to $\omega_n \sqrt{1 - 2\zeta^2}$ and exact resonant amplitude, which is $(u_o)_{\omega=\omega_n} = (u_{st})_o / [2\zeta / \sqrt{1 - 2\zeta^2}]$.

Problem 4 - Solution

1. Determine response in the first impulse

The response of the system to the first impulse is the unit response of the solution of the free vibration of the undamped SDF system due to the impulse times I . As such,

$$u_1(t) = I \left[\frac{1}{m\omega_n} \sin \omega_n t \right] \quad (4.1)$$

2. Determine response to the second impulse

$$u_2(t) = I \left[\frac{1}{m\omega_n} \sin \omega_n (t - t_d) \right], t \geq t_d \quad (4.2)$$

3. Determine response to both impulses

For $0 \leq t \leq t_d$:

$$u(t) = \frac{I}{m\omega_n} [\sin \omega_n t] = \frac{I}{m\omega_n} \left[\sin \frac{2\pi t}{T_n} \right] \quad (4.3)$$

For $t > t_d$,

$$u(t) = \frac{I}{m\omega_n} [\sin \omega_n t - \sin \omega_n (t - t_d)] \quad (4.4)$$

Hence,

$$u(t) = \frac{I}{m\omega_n} 2 \sin \left(\frac{\omega_n t_d}{2} \right) \cos \left[\frac{\omega_n (2t - t_d)}{2} \right] \quad (4.5)$$

Therefore,

$$u(t) = \frac{2I}{m\omega_n} \sin \left(\frac{\pi t_d}{T_n} \right) \cos \left[2\pi \left(\frac{t}{T_n} - \frac{1}{2} \frac{t_d}{T_n} \right) \right] \quad (4.6)$$

4. Plot displacement response

Equations (4.3) and (4.6) are plotted for $t_d/T_n = 1/8, 1/4, 1/2$, and 1 in Figures P4.1a, b, c and d, respectively.

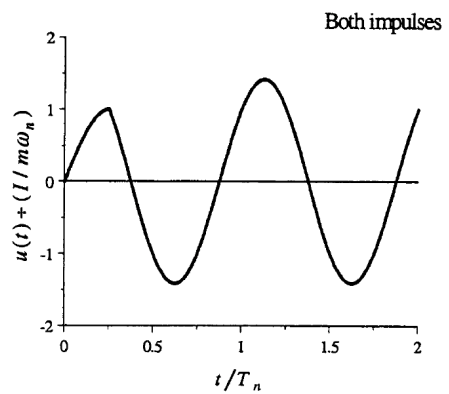
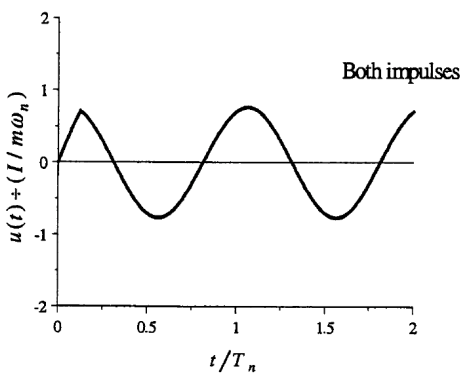
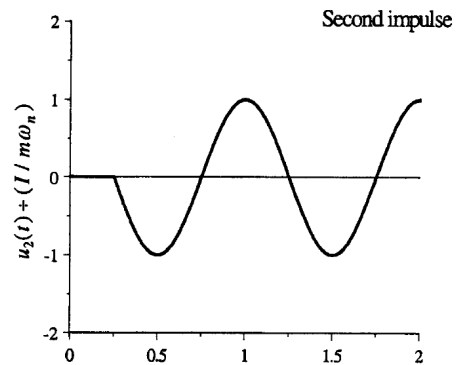
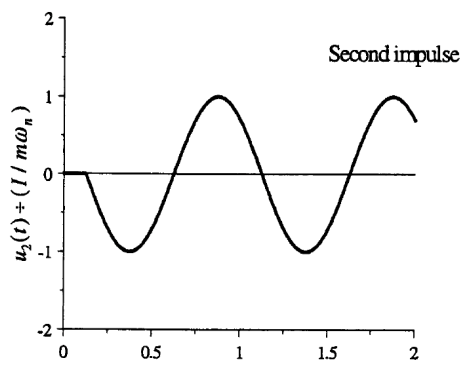
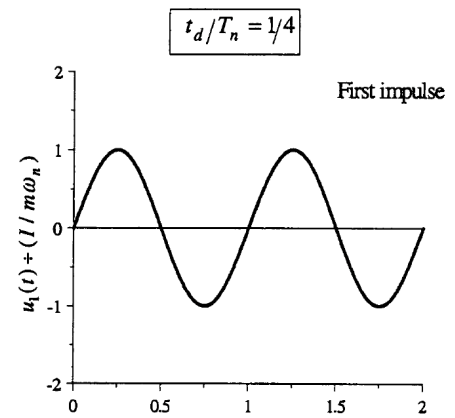
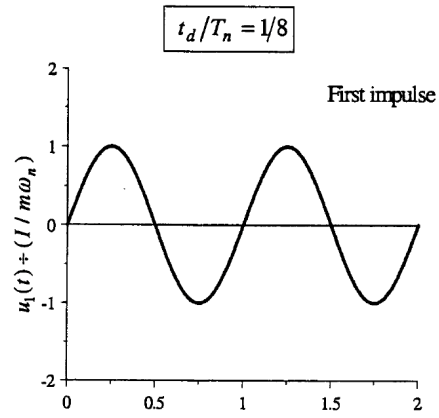


Fig. P4.3a

Fig. P4.3 b

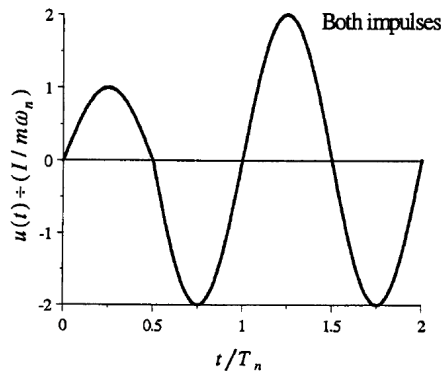
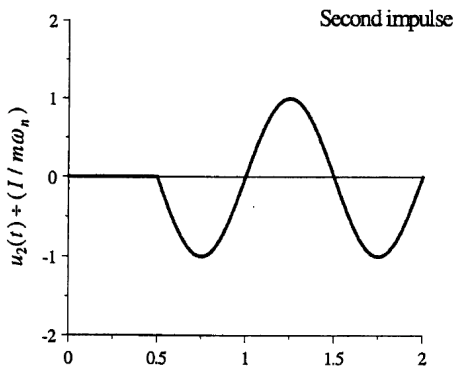
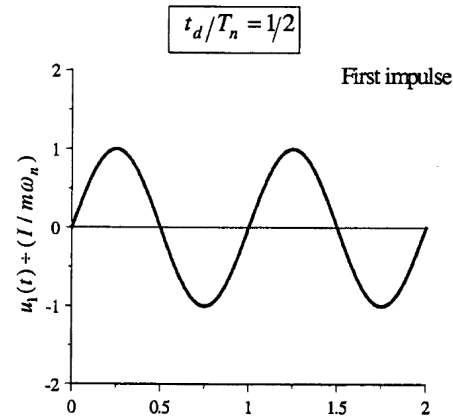


Fig. P4.3c

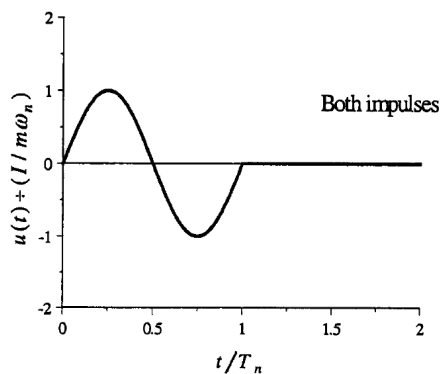
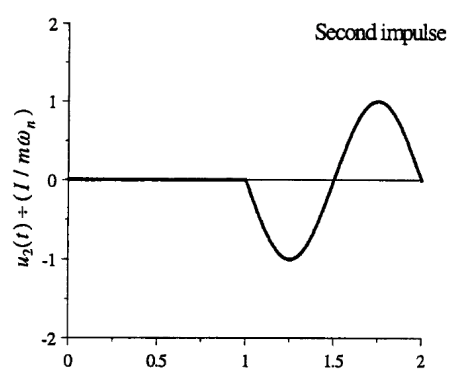
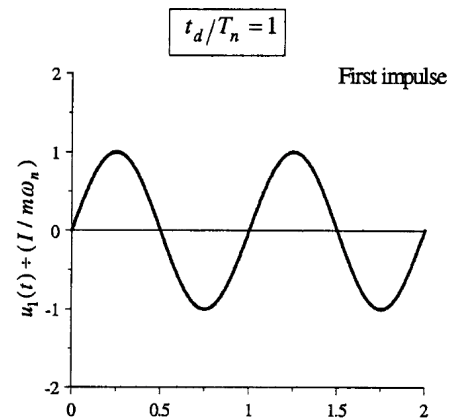


Fig. P4.3d

5. Determine maximum response during $0 \leq t \leq t_d$

The number of peaks in $u(t)$ depend on t_d/T_n ; the longer the time t_d between the pulses, more such peaks occur. The first peak occurs at $t_o = T_n/4$ with the deformation:

$$\frac{u_o}{I/m\omega_n} = 1 \quad (4.7)$$

Thus, t_d must be longer than $T_n/4$ for at least one peak to develop during $0 \leq t \leq t_d$. If t_d is shorter than $T_n/4$ no peak will develop during $0 \leq t \leq t_d$ and the response simply builds up from zero to $u(t_d)$, where,

$$\frac{u(t_d)}{I/m\omega_n} = \sin\left(\frac{2\pi t_d}{T_n}\right) \quad (4.8)$$

The maximum deformation during $0 \leq t \leq t_d$ is,

$$\frac{u_o}{I/m\omega_n} = \begin{cases} \sin(2\pi t_d/T_n), & t_d/T_n \leq 1/4 \\ 1, & t_d/T_n \geq 1/4 \end{cases} \quad (4.9)$$

Equation (4.9) is plotted in the figure below.

6. Determine maximum response during $t \geq t_d$

From Eq. (4.6), the maximum deformation during $t \geq t_d$ is,

$$\frac{u_o}{I/m\omega_n} = 2 \left| \sin\left(\frac{\pi t_d}{T_n}\right) \right| \quad (4.10)$$

Equation (4.10) is superimposed in the figure below.

