CIVIL 449: Nonlinear Analysis of Structures

School of Architecture, Civil & Environmental Engineering
Civil Engineering Institute

Integration techniques for element state determination

Prof. Dr. Dimitrios Lignos & Dr. Diego Heredia Rosa
EPFL, ENAC, IIC, RESSLab

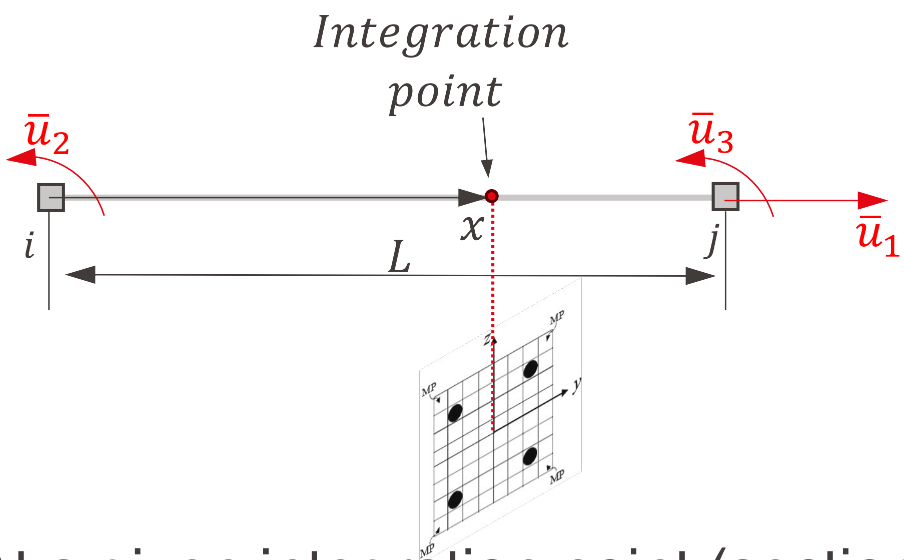
Objectives of today's lecture

To introduce:

- Fiber-based beam-column elements
- Fiber discretization of cross sections
- Constitutive models for fiber-based elements
- Computation of input strains
- Section analysis
- Type of element formulations
  - Displacement-based beam-column elements
  - Force-based beam-column elements
- Integration methods for member forces and member stiffness
    - Examples with displacement- and force-based elements

Displacement-based beam-column element

- The vectors of nodal displacements, $\bar{\mathbf{u}}$, and element resisting forces $\bar{\mathbf{q}}$, are as follows:



$$\bar{\mathbf{u}} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}^T$$

$$\bar{\mathbf{q}} = \{\bar{q}_1, \bar{q}_2, \bar{q}_3\}^T$$

- At a given integration point (section):

$$d_a(x) = N_1(x)\bar{u}_1 \qquad d_f(x) = N_2(x)\bar{u}_2 + N_3(x)\bar{u}_3$$

State determination of displacement-based element

- The tangent element stiffness matrix at iteration i of step n, $\overline{\mathbf{K}}^{n,i}$, of a displacement-based beam-column element of length L, and the element resisting force vector $\overline{\mathbf{q}}^{n,i}$ can be expressed as follows:

$$\overline{\mathbf{K}}^{n,i} = \int_0^L \overline{\mathbf{B}}^T(x) \cdot \mathbf{k}_s^{n,i}(x) \cdot \overline{\mathbf{B}}(x) \cdot dx$$

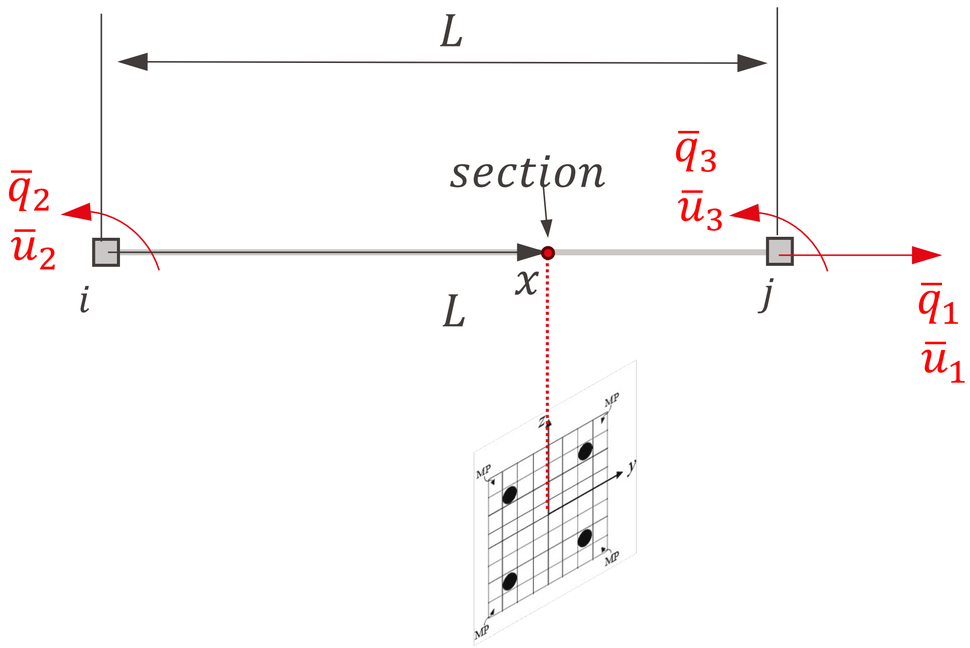
$$\overline{\mathbf{q}}^{n,i} = \int_0^L \overline{\mathbf{B}}^T(x) \cdot \mathbf{Q}_{\mathrm{sr}}^{n,i}(x) \cdot dx$$

We calculate those numerically with some numerical integration schemes

- Gauss-Legendre,
- Gauss-Lobatto,
- Gauss Radau,
- midpoint rule

Force-based (or flexibility-based) elements

- The element vector generalized nodal forces $\bar{\mathbf{q}}$ at the basic reference frame (without rigid body modes) is as follows:



$$\bar{\mathbf{q}} = \{\bar{q}_1, \bar{q}_2, \bar{q}_3\}^T$$

$$\bar{\mathbf{u}} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}^T$$

State determination of force-based element

- The section flexibility at iteration j, $\mathbf{f}_s^{n,i,j}(x)$ is,

$$\mathbf{f}_s^{n,i,j}(x) = \left(\mathbf{k}_s^{n,i,j}(x)\right)^{-1}$$

- Element flexibility matrix, $\bar{\mathbf{F}}^{n,i,j}$ at iteration j is:

$$\bar{\mathbf{F}}^{n,i,j} = \int_0^L \mathbf{b}^T(x) \cdot \mathbf{f}_s^{n,i,j}(x) \cdot \mathbf{b}(x) dx$$

- The element stiffness matrix, $\bar{\mathbf{K}}^{n,i,j}$ at iteration j is:

$$\bar{\mathbf{K}}^{n,i,j} = \left(\mathbf{F}^{n,i,j}\right)^{-1}$$

- The element end displacements at iteration j is,

$$\bar{\mathbf{u}}^{n,i,j} = \int_0^L \mathbf{b}^T(x) \cdot \mathbf{d}_s^{n,i,j}(x)\, dx$$

(end displacements) (section deformations)

Numerical integration

- In nonlinear structural mechanics, we seek to obtain numerical estimates of an integral by picking optimal coordinates (abscissas) r_i at which to evaluate the function, $f(r_i)$ of interest.
- Gauss quadrature is often used for this purpose.
- According to the theorem of Gaussian quadrature, the optimal abscissas of the m-point Gaussian quadrature formulas are precisely the roots of the orthogonal polynomial for the same interval and weighting function.
- Gauss quadrature is optimal because it fits all polynomials up to degree $2m$-1 exactly.
- Slightly less optimal fits are obtained from Radau quadrature and Laguerre-Gauss quadrature (*depends on the problem I would say!*)

Gauss quadrature

-Some remarks

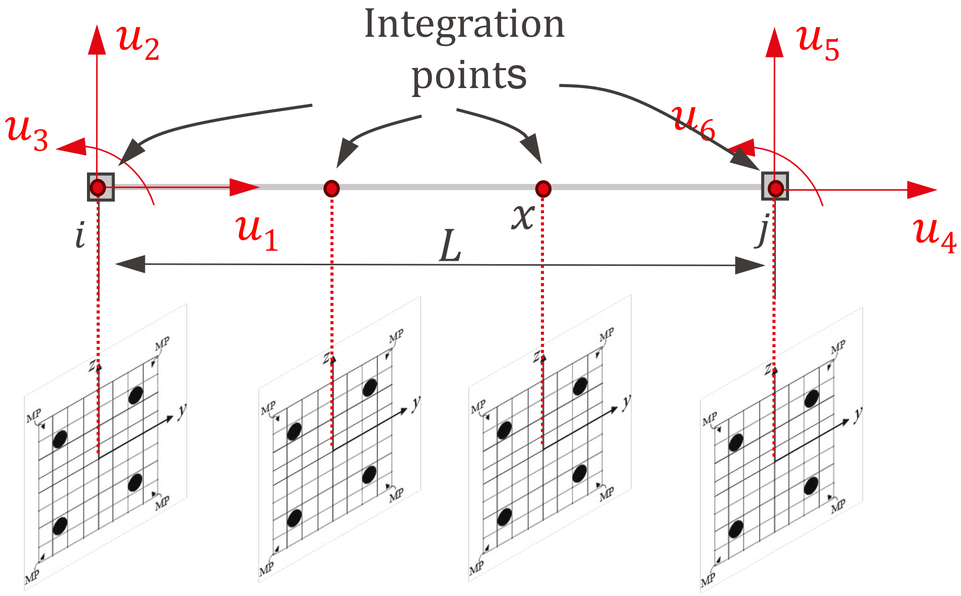
- The location of Gauss points is such that for a given number of points greatest accuracy is obtained.

- The Gauss points are located symmetrically about the center of the interval to be integrated.

- The weight will be the same for symmetrically located Gauss points about the center of the interval to be integrated.

Gauss quadrature
-One dimensional integrals

- For the state determination of beam-column elements, the integration should be done along the length L of the element. Therefore, we are dealing with one dimensional integrals.



Gauss quadrature (2)

-One dimensional integrals

- Natural coordinate system $[-1\ 1]$ instead of $[0\ L]$.

$$I = \int_{-1}^{1} f(r) \cdot dr \cong \sum_{i=1}^{n} w_i\, f_i(r_i)$$

$$f(r) = a_1 + a_2 r + a_3 r^2 + a_4 r^3$$

Gauss quadrature (3)
-One dimensional integrals

- The integral after integrating analytically,

$$I = \int_{-1}^{1} f(r) \cdot dr = \left[a_1 r + \frac{1}{2} a_2 r^2 + \frac{1}{3} a_3 r^3 + \frac{1}{4} a_4 r^4\right]_{-1}^{1} = 2a_1 + \frac{2}{3} a_3$$

Gauss quadrature (4)
-One dimensional integrals

- Assume, we would like to approximate this integral with $n = 1$ Gauss point:

$$I = \int_{-1}^{1} f(r) \cdot dr \cong w_1 f(r_1) = w_1(a_1 + a_2 r_1 + a_3 r_1^2 + a_4 r_1^3)$$

- The error then between exact and approximate solution is as follows:

$$E = a_1(2 - w_1) + a_3\left(\frac{2}{3} - r_1^2 w_1\right) - a_2 r_1 w_1 - a_4 r_1^3 w_1$$

Gauss quadrature (5)

-One dimensional integrals

- The error becomes minimum when the Jacobian $\mathbf{J}$ is zero:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\vartheta E}{\vartheta a_1} & \dfrac{\vartheta E}{\vartheta a_2} & \dfrac{\vartheta E}{\vartheta a_3} & \dfrac{\vartheta E}{\vartheta a_4} \end{bmatrix}$$

- Subsequently,

$$\mathbf{J} = \begin{bmatrix} 2 - w_1 & -w_1 r_1 & \dfrac{2}{3} - r_1^2 w_1 & -w_1 r_1^3 \end{bmatrix}$$

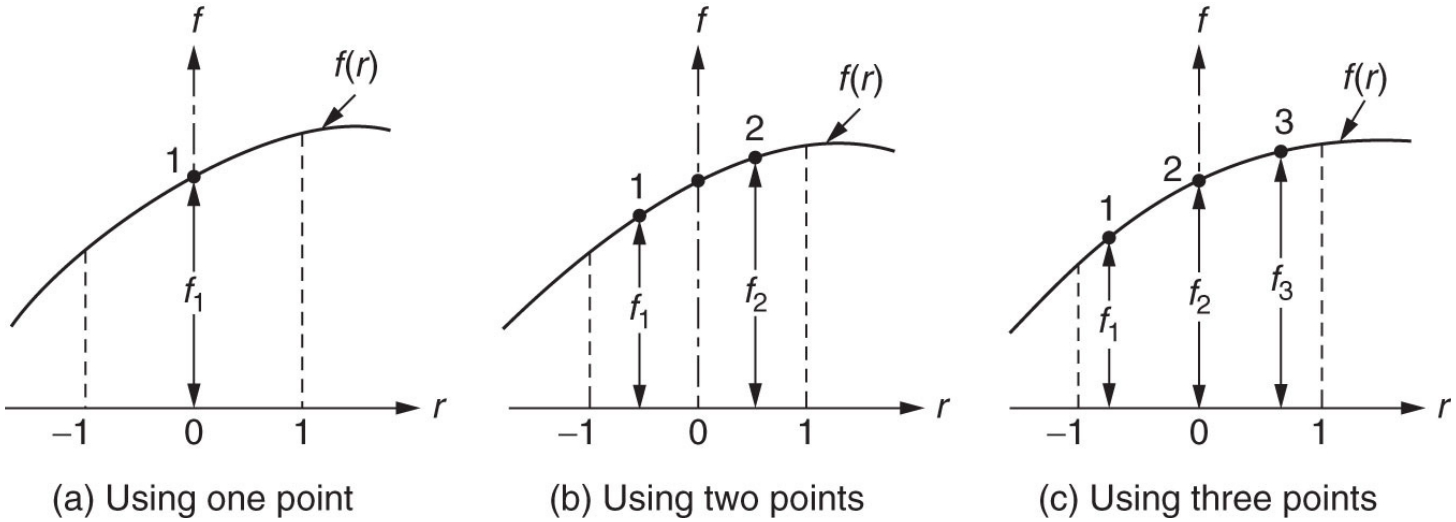
- Therefore,

$$2 - w_1 = 0 \qquad \begin{matrix} w_1 = 0 \\ r_1 = 0 \end{matrix} \qquad r_1 = \pm\sqrt{\frac{2}{3w_1}} \qquad \begin{matrix} w_1 = 0 \\ r_1^3 = 0 \end{matrix}$$

Gauss quadrature (6)
-One dimensional integrals

- Therefore, the condition that satisfies all four partial derivatives to be zero for $n = 1$ Gauss integration point is:

$$w_1 = 2 \quad \text{and} \quad r_1 = 0$$



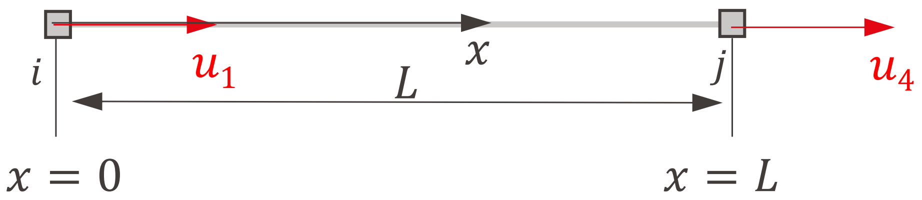
Gauss quadrature (7)

Gauss Points n	*Point* r_i	*Weight Coefficient* w_i	Polynomial Order m
1	0	2	1
2	-1/√3, 1/√3	1, 1	3
3	-√0.6, 0, √0.6	5/9, 8/9, 5/9	5
4	-0.861136, -0.339981, 0.339981, 0.861136	0.347855, 0.652145, 0.652145, 0.347855	7
5	-0.906180, -0.538469, 0, 0.538469, 0.906180	0.236927, 0.478629, 0.568889, 0.478629, 0.236927	9
6	-0.932470, -0.661209, -0.238619, 0.238619, 0.661209, 0.932470	0.171324, 0.360762, 0.467914, 0.467914, 0.360762, 0.171324	11

Gauss quadrature

-Example elastic truss element with uniform cross section

Cartesian system



Natural coordinate system

$$\mathbf{k}^e = \int_0^L \mathbf{B}^T(x) \cdot \mathbf{k}^s(x) \cdot \mathbf{B}(x) \cdot dx$$

$$= \int_0^L \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \cdot EA \cdot \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \cdot dx \qquad = \frac{1}{L^2} \int_0^L \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot EA \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot dx$$

Gauss quadrature (2)

-Example elastic truss element with uniform cross section

$$\mathbf{k}^{e} = \frac{1}{L^2}\int_0^L \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot EA \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot dx$$

- From calculus, coordinate transformation from 0 to L: to -1 to 1:
- Assume:

$$x = \frac{b-a}{2} r + \frac{b-a}{2} \qquad x = \frac{L-0}{2} r + \frac{L-0}{2}$$

$$dx = \frac{b-a}{2} dr \qquad dx = \frac{L-0}{2} dr$$

Gauss quadrature (3)

-Example elastic truss element with uniform cross section

- Therefore, $\mathbf{k}^e$ in the updated coordinate system becomes:

$$\boldsymbol{k}^e = \frac{1}{L^2}\int_{-1}^{1}\begin{bmatrix}-1\\1\end{bmatrix}\cdot EA\cdot\begin{bmatrix}-1 & 1\end{bmatrix}\cdot\frac{L}{2}dr = \frac{EA}{2L}\int_{-1}^{1}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}dr$$

$$= \frac{EA}{2L}\int_{-1}^{1}\begin{bmatrix}f_1(r) & f_2(r)\\f_2(r) & f_1(r)\end{bmatrix}dr = \frac{EA}{2L}\begin{bmatrix}2f_1(0) & 2f_2(0)\\2f_2(0) & 2f_1(0)\end{bmatrix}$$

Gauss point

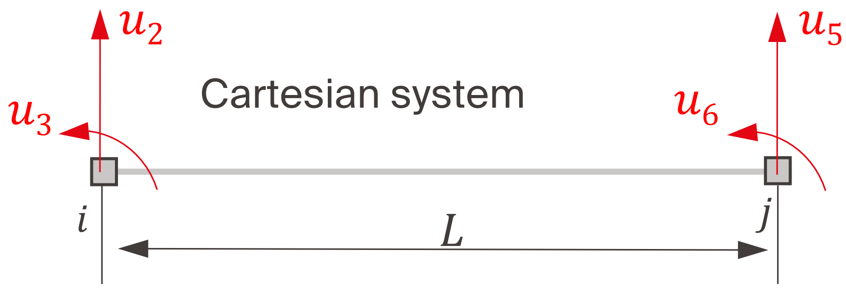
weight

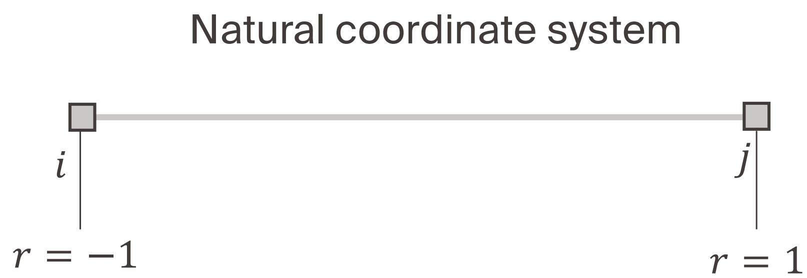
$$= \frac{EA}{2L}\begin{bmatrix}2f_1(0) & 2f_2(0)\\2f_2(0) & 2f_1(0)\end{bmatrix} = \frac{EA}{2L}\begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix} = \frac{EA}{L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

- Note: for $m = 1$ order polynomial, $n = 2m - 1 = 1$ Gauss points give an exact solution

Gauss quadrature

-Example elastic beam-column element with uniform cross section





$$\mathbf{k}^{e} = \int_{0}^{L} \mathbf{B}^{T}(x) \cdot \mathbf{k}^{s}(x) \cdot \mathbf{B}(x) \cdot dx$$

$$\mathbf{K}^{e} = \int_{0}^{L} \begin{Bmatrix} \frac{6}{L^2}\left(1 - \frac{2x}{L}\right) \\ \frac{2}{L}\left(\frac{3x}{L} - 1\right) \\ \frac{6}{L^2}\left(\frac{2x}{L} - 1\right) \\ \frac{2}{L}\left(\frac{3x}{L} - 2\right) \end{Bmatrix} \cdot EI \cdot \left[\frac{6}{L^2}\left(1 - \frac{2x}{L}\right) \quad \frac{2}{L}\left(\frac{3x}{L} - 1\right) \quad \frac{6}{L^2}\left(\frac{2x}{L} - 1\right) \quad \frac{2}{L}\left(\frac{3x}{L} - 2\right)\right] dx$$

Gauss quadrature (2)

-Example elastic beam-column element with uniform cross section

$$\mathbf{k}^{e} = \frac{4EI}{L^2}\int_0^L \begin{Bmatrix} \frac{3}{L}\left(1-\frac{2x}{L}\right) \\ \left(\frac{3x}{L}-1\right) \\ \frac{3}{L}\left(\frac{2x}{L}-1\right) \\ \left(\frac{3x}{L}-2\right) \end{Bmatrix} \cdot \begin{bmatrix} \frac{3}{L}\left(1-\frac{2x}{L}\right) & \left(\frac{3x}{L}-1\right) & \frac{3}{L}\left(\frac{2x}{L}-1\right) & \left(\frac{3x}{L}-2\right) \end{bmatrix} dx$$

$$x = \frac{L}{2}r + \frac{L}{2} \qquad dx = \frac{L}{2}dr$$

$$= \frac{2EI}{L}\int_{-1}^{1} \begin{Bmatrix} -\frac{3}{L}r \\ \frac{1}{2}(3r+1) \\ \frac{3}{L}r \\ \frac{1}{2}(3r-1) \end{Bmatrix} \cdot \begin{bmatrix} -\frac{3}{L}r & \frac{1}{2}(3r+1) & \frac{3}{L}r & \frac{1}{2}(3r-1) \end{bmatrix} dr = \frac{2EI}{L}\int_{-1}^{1} \begin{bmatrix} \frac{9}{L^2}r^2 & -\frac{3}{2L}r(3r+1) & -\frac{9}{L^2}r^2 & -\frac{3}{2L}r(3r-1) \\ -\frac{3}{2L}r(3r+1) & \frac{1}{4}(3r+1)^2 & \frac{3}{2L}r(3r+1) & \frac{1}{4}(9r^2-1) \\ -\frac{9}{L^2}r^2 & \frac{3}{2L}r(3r+1) & \frac{9}{L^2}r^2 & \frac{3}{2L}r(3r-1) \\ -\frac{3}{2L}r(3r-1) & \frac{1}{4}(9r^2-1) & \frac{3}{2L}r(3r-1) & \frac{1}{4}(3r-1)^2 \end{bmatrix} dr$$

Gauss quadrature (3)

-Example elastic beam-column element with uniform cross section

- Therefore, $\mathbf{k}^e$ in the natural coordinate system becomes:

$$= \frac{2EI}{L}\int_{-1}^{1}\begin{bmatrix} \frac{9}{L^2}r^2 & -\frac{3}{2L}r(3r+1) & -\frac{9}{L^2}r^2 & -\frac{3}{2L}r(3r-1) \\ -\frac{3}{2L}r(3r+1) & \frac{1}{4}(3r+1)^2 & \frac{3}{2L}r(3r+1) & \frac{1}{4}(9r^2-1) \\ -\frac{9}{L^2}r^2 & \frac{3}{2L}r(3r+1) & \frac{9}{L^2}r^2 & \frac{3}{2L}r(3r-1) \\ -\frac{3}{2L}r(3r-1) & \frac{1}{4}(9r^2-1) & \frac{3}{2L}r(3r-1) & \frac{1}{4}(3r-1)^2 \end{bmatrix} dr = \frac{2EI}{L}\int_{-1}^{1}\begin{bmatrix} f_1(r) & f_2(r) & f_3(r) & f_4(r) \\ & f_5(r) & f_6(r) & f_7(r) \\ & \text{Sym.} & f_8(r) & f_9(r) \\ & & & f_{10}(r) \end{bmatrix} dr$$

Gauss quadrature (4)

-Example elastic beam-column element with uniform cross section

- Assume two Gauss points:

$$= \frac{2EI}{L}\int_{-1}^{1}\begin{bmatrix} f_1(r) & f_2(r) & f_3(r) & f_4(r) \\ & f_5(r) & f_6(r) & f_7(r) \\ & \text{Sym.} & f_8(r) & f_9(r) \\ & & & f_{10}(r) \end{bmatrix} dr$$

$$\frac{2EI}{L}(f_1(-0.57735) + f_1(+0.57735))$$

$$= \frac{2EI}{L}(\frac{9}{L^2}(-0.57735)^2 + \frac{9}{L^2}(0.57735)^2)$$

$$= \frac{2EI}{L}\begin{bmatrix} \frac{6.007999}{L^2} & -\frac{5.999994}{2L} & -\frac{6.007999}{L^2} & \frac{5.999994}{2L} \\ -\frac{5.999994}{2L} & 1.999999 & \frac{5.999994}{2L} & 0.999999 \\ -\frac{6.007999}{L^2} & \frac{5.999994}{2L} & \frac{6.007999}{L^2} & \frac{2.999997}{L} \\ \frac{5.999994}{2L} & 0.999999 & \frac{2.999997}{L} & 1.999999 \end{bmatrix}$$

$$= \frac{2EI}{L}(\frac{9}{L^2}(-0.57735)^2 + \frac{9}{L^2}(0.57735)^2)$$

$$= \frac{12.016EI}{L^3}$$

(I would expect $12EI/L^3$)

Gauss quadrature (5)

-Example elastic beam-column element with uniform cross section

- Note: the error with 2 Gauss points is due to the numerical integration approximation and the number of decimals

$$\mathbf{k}^e = \frac{2EI}{L}\begin{bmatrix} \frac{6.007999}{L^2} & -\frac{5.999994}{2L} & -\frac{6.007999}{L^2} & \frac{5.999994}{2L} \\ -\frac{5.999994}{2L} & 1.999999 & \frac{5.999994}{2L} & 0.999999 \\ -\frac{6.007999}{L^2} & \frac{5.999994}{2L} & \frac{6.007999}{L^2} & \frac{2.999997}{L} \\ \frac{5.999994}{2L} & 0.999999 & \frac{2.999997}{L} & 1.999999 \end{bmatrix}$$

Gauss Lobatto

- A Gaussian quadrature with weighting function $W(x) = 1$ in which the end points of the interval [-1, 1] are included in a total of n abscissas, given $r = n - 2$ free abscissas.
- The abscissas are symmetrical about the origin.
- the general formula of integration is as follows,

$$I = \int_{-1}^{1} f(r) \cdot dr \cong w_1 f(-1) + w_n f(1) + \sum_{i=2}^{n-1} w_i f(r_i)$$

Gauss Lobatto (2)

Gauss Lobatto Points n	*Point* r_i	*Weight Coefficient* w_i	Polynomial Order m
3	$-1, 0, 1$	$\frac{1}{3}, \frac{4}{3}, \frac{1}{3}$	3
4	$-1, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 1$	$\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{6}$	5
5	$-1, -\sqrt{\frac{3}{7}}, 0, \sqrt{\frac{3}{7}}, 1$	$\frac{1}{10}, \frac{49}{90}, \frac{32}{45}, \frac{49}{90}, \frac{1}{10}$	7
6	$-1, -0.765055, -0.285232,$ $0.285232, 0.765055, 1$	$0.066667, 0.378475, 0.554858,$ $0.554858, 0.378475, 0.066667$	9

Gauss Radau

- A Gaussian quadrature-like formula for numerical estimation of integrals. It requires $m+1$ points and fits all polynomials to degree $2m$, so it effectively fits exactly all polynomials of degree $2m-1$.
- It uses a weighting function $W(x)=1$ in which the endpoint -1 in the interval [-1, 1] is included in a total of n abscissas, giving $r=n-1$ free coordinates.

The general formula of integration is as follows,

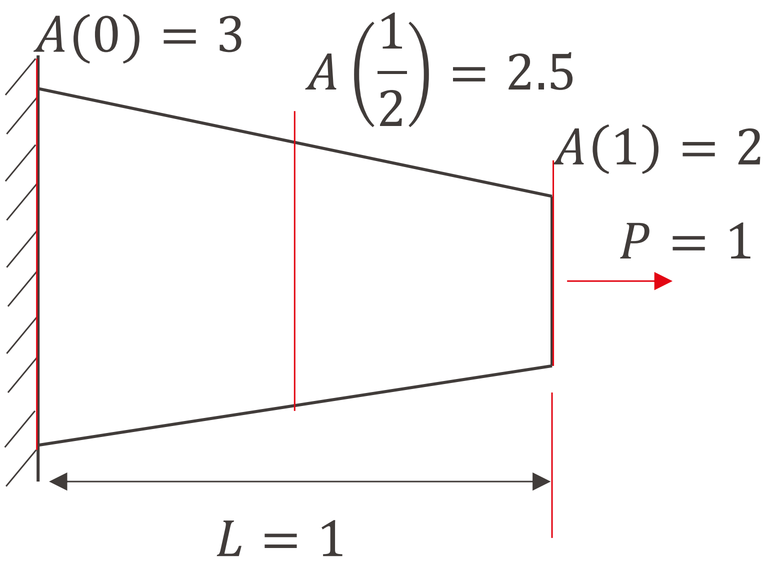
$$I=\int_{-1}^{1} f(r)\cdot dr \cong w_1 f(-1)+\sum_{i=2}^{n} w_i f(r_i)$$

Gauss Radau (2)

Gauss Lobatto Points n	*Point* r_i	*Weight Coefficient* w_i	Polynomial Order m
2	$-1, 0.333333$	$0.5, 1.5$	2
3	$-1, -0.289898, 0.689898$	$0.222222, 1.02497, 0.752806$	4
4	$-1, -0.575319, 0.181066, 0.822824$	$0.125, 0.657689, 0.776387, 0.440924$	6
5	$-1, -0.72048, -0.167181, 0.446314, 0.885792$	$0.08, 0.446208, 0.623653, 0.562712, 0.287427$	8

Example: Tapered member

- Element with linearly varying cross-section with unit length
- Subjected to unit axial load only ($P = 1$)
- Assume all units to be consistent (omitted for the sake of the example)
- Stress-strain relation is assumed as follows:



$$\sigma = \begin{cases} \varepsilon - 0.5\varepsilon^2, & \varepsilon \leq 0.95 \\ \sigma_o + 0.05(\varepsilon - 0.95), & \varepsilon > 0.95 \end{cases}$$

$$\sigma_o = \sigma[\varepsilon = 0.95] = 0.49875$$

Example: Tapered member (2)

- Let us assume 3 integration points because of the tapered cross-section

$$I = \int_{-1}^{1} f(r) \cdot dr \cong w_1 f(-1) + w_3 f(1) + w_2 f(0)$$

Integration points and weights at natural coordinate system

$$r_1 = -1 \qquad w_1 = w_3 = \frac{1}{3}$$

$$r_3 = 1 \qquad w_2 = \frac{4}{3}$$

$$r_2 = 0$$

Integration points at Cartesian system

$$x_1 = \frac{1}{2} r_1 + \frac{1}{2} = \frac{1}{2}(-1) + \frac{1}{2} = 0$$

$$x_2 = \frac{1}{2} r_2 + \frac{1}{2} = \frac{1}{2}(0) + \frac{1}{2} = \frac{1}{2}$$

$$x_3 = \frac{1}{2} r_3 + \frac{1}{2} = \frac{1}{2}(1) + \frac{1}{2} = 1$$

Corresponding area at integration points

$$A_1 = 3$$

$$A_2 = 2.5$$

$$A_3 = 2$$

Example: Tapered member (3)

- This analysis consists in a single step ($n = 1$) in load control
- The first step is to compute the structure initial stiffness matrix
- Since the member is subjected to axial load only, one fiber is enough for such computations (uniaxial loading)

Example: Tapered member (4)

- Displacement-based element: Initial element stiffness matrix

- When the tapered member is modeled with a single (axial load only) displacement-based beam-column element
- The axial displacement field can be computed as follows: $u(x) \cong \bar{N}(x)\bar{u}_1 = \frac{x}{L}\bar{u}_1$
- Therefore, the strain: $\varepsilon(x) \cong \bar{\mathrm{B}}(x)\bar{u}_1 = \frac{1}{L}\bar{u}_1$
- The initial structure tangent stiffness $\mathrm{K}^0_{structure}$ corresponds to the initial element tangent stiffness K^0,and is given as follows at iteration i:

$$\mathrm{K}^0_{structure} = \mathrm{K}^0 = \int_0^L \left(\frac{1}{L}\right)^2 \mathrm{k}^0_s(x)dx$$ (k^0_s is the initial section tangent stiffness)

$$= \int_{-1}^{1} \left(\frac{1}{L}\right)^2 \mathrm{k}^0_s\left(\frac{L}{2}r + \frac{L}{2}\right)\frac{L}{2}dr = \frac{1}{2L}\int_{-1}^{1} \underbrace{\mathrm{k}^0_s\left(\frac{L}{2}r + \frac{L}{2}\right)}_{f(r)}dr$$

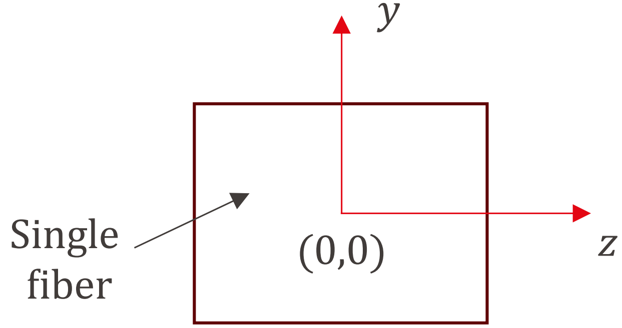
Example: Tapered member (5)

- Displacement-based element: Initial element stiffness matrix

- The section stiffness matrix k_s ,

$$\mathrm{k}_s\ (x) = \sum_{k=1}^{n_{nfib}} \mathbf{l}_{k.fiber}^T \cdot (E_{k.fiber} A_{k.fiber}) \cdot \mathbf{l}_{k.fiber}$$

$$\mathbf{l}_{k.fiber} = \{1, -y_{k.fiber}, z_{k.fiber}\}$$



$$\mathrm{k}_s\ (x) = E_{fiber}(x) \cdot A_{fiber}(x)$$

$$= \frac{d\sigma}{d\varepsilon} \cdot A_{fiber}(x)$$

(We assume one fiber in this case, therefore: $y_{k.fiber} = z_{k.fiber} = 0$)

Example: Tapered member (6)

- Displacement-based element: Initial element stiffness matrix

- Hence, after applying Gauss-Lobatto integration:

$$\mathrm{K}^0_{structure} = \mathrm{K}^0 = \frac{1}{2L}\int_{-1}^{1} f(r)dr \cong \frac{1}{2L}\left(w_1 f(r_1) + w_3 f(r_3) + w_2 f(r_2)\right)$$

$$f(r_1) = f(-1) = \mathrm{k}^0_s\left(\frac{L}{2}(-1) + \frac{L}{2}\right) = \mathrm{k}^0_s(0) = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0} \cdot A_1 = 1 \cdot A_1 = 3$$

$$f(r_2) = f(0) = \mathrm{k}^0_s\left(\frac{L}{2}(0) + \frac{L}{2}\right) = \mathrm{k}^0_s\left(\frac{L}{2}\right) = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0} \cdot A_2 = 1 \cdot A_2 = 2.5$$

$$f(r_3) = f(1) = \mathrm{k}^0_s\left(\frac{L}{2}(1) + \frac{L}{2}\right) = \mathrm{k}^0_s(L) = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0} \cdot A_3 = 1 \cdot A_3 = 2$$

Example: Tapered member (7)
- Displacement-based element: Initial element stiffness matrix

- Hence, after applying Gauss-Lobatto integration:

$$\mathrm{K}^0_{structure} = \mathrm{K}^0 = \frac{1}{2L}\left(\frac{1}{3}\cdot 3 + \frac{1}{3}\cdot 2 + \frac{4}{3}\cdot 2.5\right) = \frac{2.5}{L} = 2.5\ (L = 1)$$

Example: Tapered member (8)

- Displacement-based element: Structure state determination step

- Now that the initial structure tangent stiffness matrix was computed, the first iteration of the structure state determination can be performed

- Load-control is used, and the external load $P = 1 = \lambda_{tot} F_{ext} = 1 \cdot 1$ is applied in a single load increment ($n = 1$). Hence, the load multiplier is given by $\bar{\lambda}^1 = 1$

- For the first iteration $i = 1$ of load control, the following quantities are computed:

$$\delta \mathrm{v}_r^{1,1} = \frac{F_{unb}^0}{K_{structure}^0} = \frac{0}{K_{structure}^0} = 0 \text{ and } \delta \mathrm{v}_p^{1,1} = \frac{F_{ext}}{K_{structure}^0} = \frac{1}{2.5} = 0.4$$

- The increment in load multiplier is therefore $\delta\lambda^{1,1} = \bar{\lambda}^1 = 1$
- The increment in the structure's displacement is $\delta \mathrm{v}^{1,1} = \delta \mathrm{v}_r^{1,1} + \delta\lambda^{1,1} \delta \mathrm{v}_p^{1,1} = 0.4$

Example: Tapered member (9)

- Displacement-based element: Element state determination step

- The increment in the element displacement is $\Delta\bar{\mathrm{u}}^{1,1} = \delta\mathrm{v}^{1,1} = 0.4$
- The element displacement is therefore $\bar{\mathrm{u}}^{1,1} = \bar{\mathrm{u}}^{1,0} + \Delta\bar{\mathrm{u}}^{1,1} = 0 + 0.4 = 0.4$
- The section deformations at the three integration sections can be computed:

$$\mathrm{d}_s^{1,1}(x) = \bar{\mathrm{B}}(x)\bar{\mathrm{u}}^{1,1}$$

$$\mathrm{d}_s^{1,1}(x = 0) = \frac{1}{L}\bar{\mathrm{u}}^{1,1} = \frac{1}{1} \cdot 0.4 = 0.4$$

$$\mathrm{d}_s^{1,1}(x = L/2) = \frac{1}{L}\bar{\mathrm{u}}^{1,1} = \frac{1}{1} \cdot 0.4 = 0.4$$

$$\mathrm{d}_s^{1,1}(x = L) = \frac{1}{L}\bar{\mathrm{u}}^{1,1} = \frac{1}{1} \cdot 0.4 = 0.4$$

Example: Tapered member (10)

- Displacement-based element: Section state determination step

- For every section along the element length, the strain at each fiber is computed using

$$\boldsymbol{\varepsilon}_{k.fiber} = \mathbf{l}_{k.fiber} \cdot \boldsymbol{d}_{s}^{1,1} \rightarrow \varepsilon_{k.fiber} = 1 \cdot d_{s}^{1,1}$$

- Recall that each section is composed of a single fiber, therefore:

$$\varepsilon_{k.fiber}^{1,1}(x = 0) = 1 \cdot 0.4 = 0.4$$

$$\varepsilon_{k.fiber}^{1,1}(x = L/2) = 1 \cdot 0.4 = 0.4$$

$$\varepsilon_{k.fiber}^{1,1}(x = L) = 1 \cdot 0.4 = 0.4$$

Example: Tapered member (11)

- Displacement-based element: Section state determination step

- For every fiber, the stress and material tangent stiffness can be determined:

$$\sigma^{1,1}(x=0)=\varepsilon^{1,1}(0)-0.5\varepsilon^{1,1}(0)^2=0.4-0.5\cdot0.4^2=0.32$$

$$\sigma^{1,1}(x=L/2)=\varepsilon^{1,1}(L/2)-0.5\varepsilon^{1,1}(L/2)^2=0.4-0.5\cdot0.4^2=0.32$$

$$\sigma^{1,1}(x=L)=\varepsilon^{1,1}(L)-0.5\varepsilon^{1,1}(L)^2=0.4-0.5\cdot0.4^2=0.32$$

$$\left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0.4}=1-0.4=0.6$$

Example: Tapered member (12)

- Displacement-based element: Section state determination step

- For every section, the section resisting force can be determined:

$$Q_{sr}^{1,1}(x) = \sum_{fibers} \sigma^{1,1}(x) \cdot A(x)$$

$$Q_{sr}^{1,1}(x = 0) = \sigma^{1,1}(x = 0) \cdot A(x = 0) = 0.32 \cdot 3 = 0.96$$

$$Q_{sr}^{1,1}(x = L/2) = \sigma^{1,1}(x = L/2) \cdot A(x = L/2) = 0.32 \cdot 2.5 = 0.80$$

$$Q_{sr}^{1,1}(x = L) = \sigma^{1,1}(x = L) \cdot A(x = L) = 0.32 \cdot 2 = 0.64$$

Example: Tapered member (13)

- Displacement-based element: Section state determination step

- Similarly, the section tangent stiffness can be determined:

$$\mathrm{k}_S^{1,1}(x) = \sum_{fibers} \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon}(x) \cdot A(x)$$

$$\mathrm{k}_S^{1,1}(x=0) = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0.4}(x=0) \cdot A(x=0) = 0.6 \cdot 3 = 1.8$$

$$\mathrm{k}_S^{1,1}(x=L/2) = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0.4}(x=L/2) \cdot A(x=L/2) = 0.6 \cdot 2.5 = 1.5$$

$$\mathrm{k}_S^{1,1}(x=L) = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0.4}(x=L) \cdot A(x=L) = 0.6 \cdot 2 = 1.2$$

Example: Tapered member (14)

- Displacement-based element: Element state determination step

- The element resisting force can be determined:

$$\mathrm{Q}^{1,1} = \bar{q}^{1,1} \approx \frac{L}{2} \cdot \frac{1}{L} \cdot \left(w_1 \mathrm{Q}_{sr}^{1,1}(x=0) + w_3 \mathrm{Q}_{sr}^{1,1}(x=L/2) + w_2 \mathrm{Q}_{sr}^{1,1}(x=L) \right)$$

$$= \frac{1}{2} \cdot \left(\frac{1}{3} \cdot 0.96 + \frac{4}{3} \cdot 0.8 + \frac{1}{3} \cdot 0.64 \right) = 0.8$$

- Similarly, the element tangent stiffness can be determined:

$$\mathrm{K}^{1,1} = \overline{\mathrm{K}}^{1,1} \cong \frac{1}{2L} \left(w_1 \mathrm{k}_s^{1,1}(x=0) + w_3 \mathrm{k}_s^{1,1}(x=L/2) + w_2 \mathrm{k}_s^{1,1}(x=L) \right)$$

$$= \frac{1}{2 \cdot 1} \left(\frac{1}{3} \cdot 1.8 + \frac{4}{3} \cdot 1.5 + \frac{1}{3} \cdot 1.2 \right) = 1.5$$

Example: Tapered member (15)

- Displacement-based element: Structure state determination step

- The structure internal force is therefore

$$\mathrm{F}_{int}^{1,1} = Q^{1,1} = 0.8$$

- The element unbalance force is given by

$$\mathrm{F}_{unb}^{1,1} = \mathrm{F}_{int}^{1,1} - \lambda^{1,1} \cdot \mathrm{F}_{ext} = 0.8 - 1 \cdot 1 = -0.2$$

- At next iteration $(i = 2)$ of the Newton-Raphson algorithm for the load-control integrator, this unbalance force is used to compute the increment in the structure displacements

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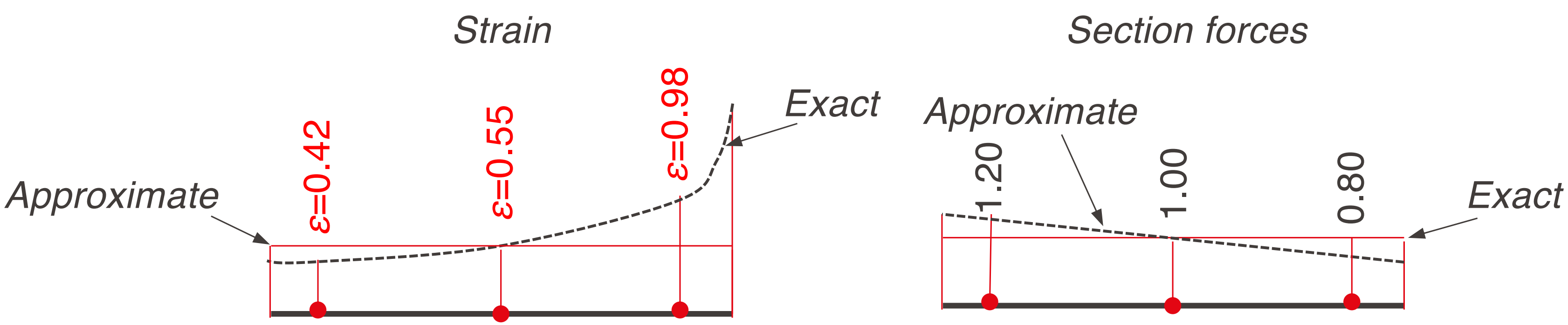
Example: Tapered member (16)

- Displacement-based element: Results iterations

i	u	Section	ε	σ	$d\sigma/d\varepsilon$	Q_{sr}	k_s	Q	K	F_{unb}
1	0.40	1	0.40	0.32	0.60	0.96	1.80	0.80	1.50	-0.20
		2	0.40	0.32	0.60	0.80	1.50			
		3	0.40	0.32	0.60	0.64	1.20			
2	0.53	1	0.53	0.39	0.47	1.17	1.40	0.98	1.17	-0.02
		2	0.53	0.39	0.47	0.98	1.17			
		3	0.53	0.39	0.47	078	0.93			
3	0.55	1	0.55	0.40	0.45	1.20	1.34	1.00	1.12	0
		2	0.55	0.40	0.45	1.00	1.12			
		3	0.55	0.40	0.45	0.80	0.90			

Example: Tapered member (17)

-Some important remarks



- The strain along the member remains constant, which is not correct due to the approximate nature of the axial displacement interpolation.
- The corresponding section forces are not correct along the member.
- Satisfies equilibrium in the weighted residual sense, it does not satisfy equilibrium in a strict point-by-point sense.
- Force-based beam-column elements should yield the exact answer.

Example: Tapered member (18)

- Force-based element: Initial element stiffness matrix

- The initial element flexibility is given as follows:

$$F^0 = \int_0^L \mathbf{b}^T(x) \cdot \left(\mathbf{k}_s^0(x)\right)^{-1} \cdot \mathbf{b}(x)dx = \int_0^L 1 \cdot f_s^0(x) \cdot 1dx \quad \left(\mathbf{b}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{x}{L} - 1\right) & \left(\frac{x}{L}\right) \end{bmatrix}\right)$$

$$= \int_{-1}^{1} f_s^0\left(\frac{L}{2}r + \frac{L}{2}\right)\frac{L}{2}dr = \frac{L}{2}\int_{-1}^{1} \underbrace{f_s^0\left(\frac{L}{2}r + \frac{L}{2}\right)}_{g(r)} dr$$

Example: Tapered member (19)

- Force-based element: Initial element stiffness matrix

- With (see slide 34):

$$\mathrm{f}_s^0(0) = \frac{1}{\mathrm{k}_s^0(0)} = \frac{1}{3} = 0.33$$

$$\mathrm{f}_s^0(L/2) = \frac{1}{\mathrm{k}_s^0(L/2)} = \frac{1}{2.5} = 0.40$$

$$\mathrm{f}_s^0(L) = \frac{1}{\mathrm{k}_s^0(L)} = \frac{1}{2} = 0.50$$

- With the selected numerical integration rule:

$$\mathrm{F}^0 = \overline{\mathrm{F}}^0 = \frac{L}{2} \cdot \left(w_1 \mathrm{f}_s^0(0) + w_2 \mathrm{f}_s^0(L/2) + w_3 \mathrm{f}_s^0(L) \right)$$

$$= \frac{1}{2} \cdot \left(\frac{1}{3} \cdot 0.33 + \frac{4}{3} \cdot 0.40 + \frac{1}{3} \cdot 0.50 \right) = 0.41$$

Example: Tapered member (20)

- Force-based element: Initial element stiffness matrix

- The initial element stiffness (and therefore the initial structure stiffness) is given by

$$\mathrm{K}^0_{structure} = \overline{\mathrm{K}}^0 = \mathrm{K}^0 = \frac{1}{\mathrm{F}^0} = \frac{1}{0.41} = 2.47$$

Example: Tapered member (21)

- Force-based element: Structure state determination step

- The initial structure tangent stiffness matrix was computed; therefore, the first iteration of the structure state determination can be performed

- Load-control is used, and the external load $P = 1 = \lambda_{tot} F_{ext} = 1 \cdot 1$ is applied in a single load increment ($n = 1$). Hence, the load multiplier is given by $\bar{\lambda}^1 = 1$

- For the first iteration $i = 1$ of load control, the following quantities are computed:

$$\delta \mathrm{v}_r^{1,1} = \frac{F_{unb}^0}{K_{structure}^0} = \frac{0}{K_{structure}^0} = 0 \text{ and } \delta \mathrm{v}_p^{1,1} = \frac{F_{ext}}{K_{structure}^0} = \frac{1}{2.47} = 0.406$$

- The increment in load multiplier is therefore $\delta\lambda^{1,1} = \bar{\lambda}^1 = 1$
- The increment in the structure's displacement is $\delta \mathrm{v}^{1,1} = \delta \mathrm{v}_r^{1,1} + \delta\lambda^{1,1} \delta \mathrm{v}_p^{1,1} = 0.406$

Example: Tapered member (22)

- Force-based element: Element state determination step

- The increment in the element displacement is $\Delta\bar{\mathrm{u}}^{1,1} = \delta \mathrm{v}^{1,1} = 0.406$
- The element displacement is therefore $\bar{\mathrm{u}}^{1,1} = \bar{\mathrm{u}}^{1,0} + \Delta\bar{\mathrm{u}}^{1,1} = 0 + 0.406 = 0.406$
- The Newton-Raphson procedure to ensure convergence of the element state determination step of the force-based beam-column element is started ($j = 1$)
- The element force increment is computed:

$$\Delta\bar{\mathrm{q}}^{1,1,1} = \bar{K}^{1,1,1}\Delta\bar{\mathrm{u}}^{1,1,1} = 2.47 \cdot 0.406 = 1.0$$

- The element force is updated:

$$\bar{\mathrm{q}}^{1,1,1} = \bar{\mathrm{q}}^{1,0,1} + \Delta\bar{\mathrm{q}}^{1,1,1} = 0 + 1.0 = 1.0$$

Example: Tapered member (23)

- Force-based element: Element state determination step

- The increment in section force is computed at each section using

$$\Delta Q_s^{1,1,1}(x) = b(x)\Delta\bar{\mathrm{q}}^{1,1,1}$$

This gives:

$$\Delta Q_s^{1,1,1}(0) = 1.0 \cdot 1.0 = 1.0$$
$$\Delta Q_s^{1,1,1}(L/2) = 1.0 \cdot 1.0 = 1.0$$
$$\Delta Q_s^{1,1,1}(L) = 1.0 \cdot 1.0 = 1.0$$

- The section force is updated:

$$Q_s^{1,1,1}(x) = Q_s^{1,1,0}(x) + \Delta Q_s^{1,1,1}(x)$$

This gives:

$$Q_s^{1,1,1}(0) = 0.0 + 1.0 = 1.0$$
$$Q_s^{1,1,1}(L/2) = 0.0 + 1.0 = 1.0$$
$$Q_s^{1,1,1}(L) = 0.0 + 1.0 = 1.0$$

Example: Tapered member (24)

- Force-based element: Element state determination step

- The increment in section deformation is computed at each section using

$$\Delta d_s^{1,1,1}(x) = d_{su}^{1,1,0}(x) + \mathrm{f}_s^{1,1,0}(x) \cdot \Delta Q_s^{1,1,1}(x)$$

This gives:

$$\Delta d_s^{1,1,1}(0) = 0.0 + 0.33 \cdot 1.0 = 0.33$$
$$\Delta d_s^{1,1,1}(L/2) = 0.0 + 0.40 \cdot 1.0 = 0.40$$

$$\Delta d_s^{1,1,1}(L) = 0.0 + 0.50 \cdot 1.0 = 0.50$$

- The section deformation is updated:

$$d_s^{1,1,1}(x) = d_s^{1,1,0}(x) + \Delta d_s^{1,1,1}(x)$$

This gives:

$$d_s^{1,1,1}(0) = 0.0 + 0.33 = 0.33$$
$$d_s^{1,1,1}(L/2) = 0.0 + 0.40 = 0.40$$
$$d_s^{1,1,1}(L) = 0.0 + 0.50 = 0.50$$

Example: Tapered member (25)

- Force-based element: Section state determination step

- For every section along the element length, the strain at each fiber is computed using

$$\boldsymbol{\varepsilon}_{k.fiber} = \mathbf{l}_{k.fiber} \cdot \boldsymbol{d}_s^{1,1,1} \rightarrow \varepsilon_{k.fiber} = 1 \cdot d_s^{1,1,1}$$

- Since each section is composed of a single fiber:

$$\varepsilon_{k.fiber}^{1,1,1}(x = 0) = 1 \cdot 0.33 = 0.33$$

$$\varepsilon_{k.fiber}^{1,1,1}(x = L/2) = 1 \cdot 0.40 = 0.40$$

$$\varepsilon_{k.fiber}^{1,1,1}(x = L) = 1 \cdot 0.50 = 0.50$$

Example: Tapered member (26)

- Force-based element: Section state determination step

- For every fiber, the stress and material tangent stiffness can be determined:

$$\sigma^{1,1,1}(x=0) = \varepsilon^{1,1,1}(0) - 0.5\varepsilon^{1,1,1}(0)^2 = 0.33 - 0.5 \cdot 0.33^2 = 0.28$$
$$\sigma^{1,1,1}(x=L/2) = \varepsilon^{1,1,1}(L/2) - 0.5\varepsilon^{1,1,1}(L/2)^2 = 0.4 - 0.5 \cdot 0.4^2 = 0.32$$
$$\sigma^{1,1,1}(x=L) = \varepsilon^{1,1,1}(L) - 0.5\varepsilon^{1,1,1}(L)^2 = 0.5 - 0.5 \cdot 0.5^2 = 0.375$$

$$\left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0.33}(x=0) = 1 - 0.33 = 0.67$$
$$\left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0.4}(x=L/2) = 1 - 0.4 = 0.6$$
$$\left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0.5}(x=L) = 1 - 0.5 = 0.5$$

Example: Tapered member (27)

- Force-based element: Section state determination step

- For every section, the section resisting force can be determined:

$$Q_{sr}^{1,1,1}(x) = \sum_{fibers} \sigma^{1,1,1}(x) \cdot A(x)$$

$$Q_{sr}^{1,1,1}(x = 0) = \sigma^{1,1,1}(x = 0) \cdot A(x = 0) = 0.28 \cdot 3 = 0.83$$

$$Q_{sr}^{1,1,1}(x = L/2) = \sigma^{1,1,1}(x = L/2) \cdot A(x = L/2) = 0.32 \cdot 2.5 = 0.80$$

$$Q_{sr}^{1,1,1}(x = L) = \sigma^{1,1,1}(x = L) \cdot A(x = L) = 0.375 \cdot 2 = 0.75$$

Example: Tapered member (28)

- Force-based element: Section state determination step

- Similarly, the section tangent stiffness can be determined:

$$\mathrm{k}_s^{1,1,1}(x) = \sum_{fibers} \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon} (x) \cdot A(x)$$

$$\mathrm{k}_s^{1,1,1}(x=0) = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0.33} (x=0) \cdot A(x=0) = 0.67 \cdot 3 = 2.0$$

$$\mathrm{k}_s^{1,1,1}(x=L/2) = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0.4} (x=L/2) \cdot A(x=L/2) = 0.6 \cdot 2.5 = 1.5$$

$$\mathrm{k}_s^{1,1,1}(x=L) = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0.5} (x=L) \cdot A(x=L) = 0.5 \cdot 2 = 1.0$$

Example: Tapered member (29)

- Force-based element: Section state determination step

- The section flexibility is computed:

$$f_s^{1,1,1}(x) = \frac{1}{\mathrm{k}_s^{1,1,1}(x)}$$

$$f_s^{1,1,1}(x = 0) = \frac{1}{\mathrm{k}_s^{1,1,1}(x = 0)} = \frac{1}{2.0} = 0.5$$

$$f_s^{1,1,1}(x = L/2) = \frac{1}{\mathrm{k}_s^{1,1,1}(x = L/2)} = \frac{1}{1.5} = 0.67$$

$$f_s^{1,1,1}(x = L) = \frac{1}{\mathrm{k}_s^{1,1,1}(x = L)} = \frac{1}{1.0} = 1.0$$

Example: Tapered member (30)

- Force-based element: Element state determination step

- The section unbalanced force is computed for every section:

$$Q_{su}^{1,1,1}(x) = \mathrm{Q}_{\mathrm{s}}^{1,1,1}(x) - \mathrm{Q}_{\mathrm{sr}}^{1,1,1}(x)$$

Which gives

$$Q_{su}^{1,1,1}(0) = 1.0 - 0.83 = 0.17$$

$$Q_{su}^{1,1,1}(0) = 1.0 - 0.80 = 0.2$$

$$Q_{su}^{1,1,1}(0) = 1.0 - 0.75 = 0.25$$

Example: Tapered member (31)

- Force-based element: Element state determination step

- The section unbalanced deformation is computed for every section:

$$\mathrm{d}_{\mathrm{su}}^{1,1,1}(x) = \mathrm{f}_{s}^{1,1,1}(x) \cdot Q_{su}^{1,1,1}(x)$$

Which gives

$$\mathrm{d}_{\mathrm{su}}^{1,1,1}(0) = 0.5 \cdot 0.17 = 0.085$$

$$\mathrm{d}_{\mathrm{su}}^{1,1,1}(L/2) = 0.67 \cdot 0.20 = 0.133$$

$$\mathrm{d}_{\mathrm{su}}^{1,1,1}(L) = 1.0 \cdot 0.25 = 0.25$$

Example: Tapered member (32)

- Force-based element: Element state determination step

- The element resisting force can be determined:

$$Q^{1,1,1} = \bar{q}^{1,1,1} \approx \frac{L}{2} \cdot \left(w_1 \mathrm{Q}_{sr}^{1,1,1}(x = 0) + w_3 \mathrm{Q}_{sr}^{1,1,1}(x = L/2) + w_2 \mathrm{Q}_{sr}^{1,1,1}(x = L) \right)$$

$$= \frac{1}{2} \cdot \left(\frac{1}{3} \cdot 0.83 + \frac{4}{3} \cdot 0.8 + \frac{1}{3} \cdot 0.75 \right) = 0.80$$

- Similarly, the element flexibly can be determined:

$$F^{1,1,1} = \bar{F}^{1,1,1} \cong \frac{L}{2} \cdot \left(w_1 f_s^{1,1,1}(x = 0) + w_3 f_s^{1,1,1}(x = L/2) + w_2 f_s^{1,1,1}(x = L) \right)$$

$$= \frac{1}{2} \cdot \left(\frac{1}{3} \cdot 0.5 + \frac{4}{3} \cdot 0.67 + \frac{1}{3} \cdot 1.0 \right) = 0.69$$

Example: Tapered member (33)

- Force-based element: Element state determination step

- Therefore, the element tangent stiffness is computed:

$$\mathrm{K}^{1,1,1} = \overline{\mathrm{K}}^{1,1,1} = \frac{1}{F^{1,1,1}} = \frac{1}{0.69} = 1.44$$

- The element unbalanced displacement is computed:

$$\overline{\mathbf{u}}_{\mathrm{u}}^{1,1,1} = \int_{\mathbf{0}}^{\boldsymbol{L}} \mathbf{b}^{T}(x)\mathbf{d}_{\mathrm{su}}^{1,1,1}(x)\, dx$$

Which gives

$$\bar{\mathrm{u}}_{\mathrm{u}}^{1,1,1} = \frac{\mathrm{L}}{2} \cdot \left(w_1 d_{su}^{1,1,1}(x=0) + w_3 d_{su}^{1,1,1}(x=L/2) + w_2 d_{su}^{1,1,1}(x=L) \right)$$
$$= \frac{1}{2} \cdot \left(\frac{1}{3} \cdot 0.085 + \frac{4}{3} \cdot 0.133 + \frac{1}{3} \cdot 0.25 \right) = 0.144$$

Example: Tapered member (34)

- Force-based element: Element state determination step

- For the next iteration ($j = 2$) of the Newton-Raphson procedure for the element state determination of the force-based element, the increment in the element deformation is set to

$$\Delta\bar{\mathrm{u}}^{1,1,2} = -\bar{\mathrm{u}}_{u}^{1,1,1} = -0.144$$

- The element state determination loop in continued until the element has converged
- Then another iteration ($\mathrm{i} = 2$) of the Newton-Raphson procedure for the structure state determination is conducted following the load-control integrator

Example: Tapered member (35)

- Force-based element: Results iterations (1)

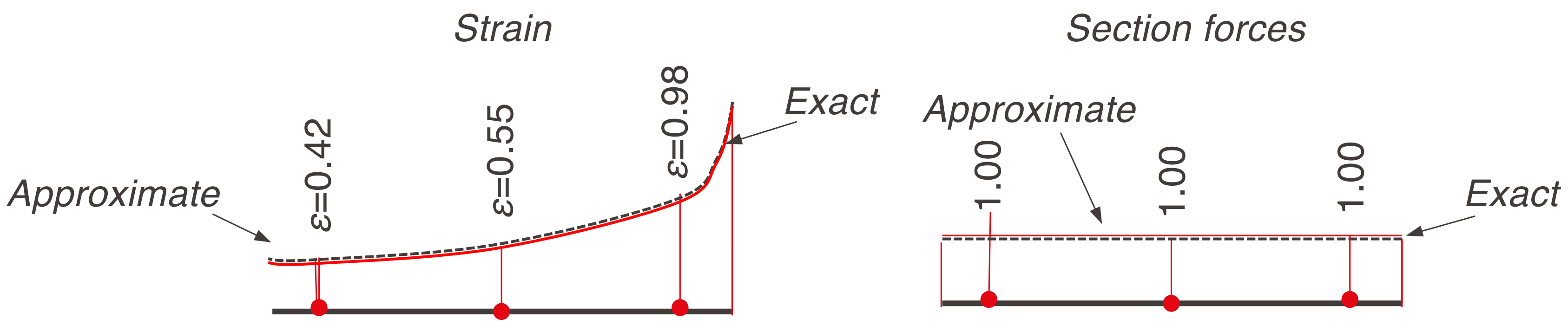
i	u	j	Sec	ε	σ	$d\sigma/d\varepsilon$	Q_{sr}	Q_{su}	d_{su}	k_s	u_u	Q	K_{struct}	F_{unb}
1	0.41	1	1	0.33	0.28	0.67	0.83	0.17	0.08	2.00	0.14	0.79	1.42	-0.21
			2	0.40	0.32	0.60	0.80	0.20	0.13	1.50				
			3	0.50	0.37	0.50	0.75	0.25	0.25	1.00				
		2	1	0.31	0.26	0.69	0.79	0.00	0.00	2.06	0.00			
			2	0.39	0.32	0.61	0.79	0.00	0.00	1.51				
			3	0.54	0.40	0.46	0.79	0.00	0.00	0.92				
2	0.55	1	1	0.41	0.33	0.59	0.98	0.02	0.01	1.76	0.03	0.97	0.91	-0.03
			2	0.53	0.39	0.47	0.98	0.02	0.02	1.17				
			3	0.77	0.47	0.23	0.95	0.05	0.11	0.46				
		2	1	0.40	0.32	0.60	0.97	0.00	0.00	1.79	0.00			
			2	0.52	0.39	0.48	0.97	0.00	0.00	1.19				
			3	0.81	0.48	0.19	0.97	0.00	0.00	0.37				
3	0.59	1	1	0.42	0.33	0.58	1.00	0.00	0.00	1.73	0.01			
			2	0.55	0.40	0.45	1.00	0.00	0.00	1.12				
			3	0.91	0.50	0.09	0.99	0.01	0.05	0.19				

Example: Tapered member (36)

- Force-based element: Results iterations (2)

i	u	j	Sec	ε	σ	$d\sigma/d\varepsilon$	Q_{sr}	Q_{su}	d_{su}	k_s	u_u	Q	K_{struct}	F_{unb}
3	0.59	2	1	0.42	0.33	0.58	0.99	0.00	0.00	1.74	0.00	0.99	0.56	-0.01
			2	0.55	0.40	0.45	0.99	0.00	0.00	1.13				
			3	0.92	0.50	0.08	0.99	0.00	0.00	0.15				
4	0.60	1	1	0.42	0.33	0.58	1.00	0.00	0.00	1.73	0.00	1,00	0.31	-0.00
			2	0.55	0.40	0.45	1.00	0.00	0.00	1.12				
			3	0.96	0.50	0.04	1.00	0.00	0.02	0.08				
		2	1	0.42	0.33	0.58	1.00	0.00	0.00	1.73	0.00			
			2	0.55	0.40	0.45	1.00	0.00	0.00	1.12				
			3	0.97	0.50	0.03	1.00	0.00	0.00	0.07				
5	0.60	1	1	0.42	0.33	0.58	1.00	0.00	0.00	1.73	0.00	1.00	0.17	0.00
			2	0.55	0.40	0.45	1.00	0.00	0.00	1.12				
			3	0.98	0.50	0.02	1.00	0.00	0.01	0.03				
		2	1	0.42	0.33	0.58	1.00	0.00	0.00	1.73	0.00			
			2	0.55	0.40	0.45	1.00	0.00	0.00	1.12				
			3	0.98	0.50	0.02	1.00	0.00	0.00	0.03				

Example: Tapered member (19)
-force-based elements



- The strain along the member is accurate
- The corresponding section forces are correct along the member
- Satisfies equilibrium in a strict point-by-point sense
- Force-based elements yield the exact answer