CIVIL 449: Nonlinear Analysis of Structures

School of Architecture, Civil & Environmental Engineering
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Enforcement of Constraints

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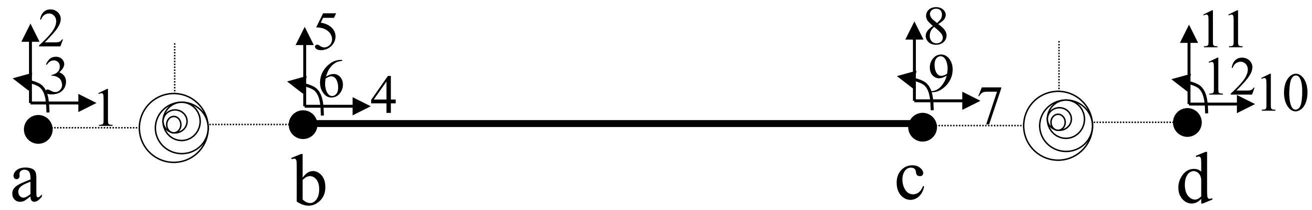
Objectives of today's lecture

- To introduce:

  - How to apply constraints in finite element models

Motivation

- The figure below shows an elastic beam-column element with two rotational springs at the element ends



- As seen in Week#3, nodes a and b have the same coordinates (same for nodes c and d)

- The left and right springs act between degrees of freedom 3 and 6 (left spring) and 9 and 12 (right spring)

- Additional constraints must be defined between degrees of freedom 1,2,4 and 5 (left spring) and 7,8,10 and 11 (right spring) to ensure that the springs remain at zero length

Additional constraints

- The following constraints are added to the model:

$$\text{Left spring:}\begin{cases} v_1 = v_4 \\ v_2 = v_5 \end{cases} \text{ and right spring:}\begin{cases} v_7 = v_{10} \\ v_8 = v_{11} \end{cases}$$

Where v_k denotes the displacement at the global degree of freedom k

Constraints enforcement – Transformation approach (1)

- Constraint equations that couple degrees of freedom in $\boldsymbol{v}$ can be written in the form

$$\boldsymbol{C} \cdot \boldsymbol{v} = \boldsymbol{Q}$$

Where $\boldsymbol{C}$ and $\boldsymbol{Q}$ contain constants.

- Consider the common case $\boldsymbol{Q} = \mathbf{0}$, the constraint equation is partitioned so that

$$\begin{bmatrix} \boldsymbol{C}_r & \boldsymbol{C}_c \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_r \\ \boldsymbol{v}_c \end{bmatrix} = \mathbf{0}$$

Where $\boldsymbol{v}_r$ and $\boldsymbol{v}_c$ are the dofs to be retained and dofs to be condensed out, respectively

- Because there are as many dofs $\boldsymbol{v}_c$ as there are independent equations of constraint, matrix $\boldsymbol{C}_c$ is square and nonsingular

Constraints enforcement – Transformation approach (2)

- Solving for $\boldsymbol{v}_c$ yields

$$\boldsymbol{v}_c = \boldsymbol{C}_{rc}\boldsymbol{v}_r \text{ where } \boldsymbol{C}_{rc} = -\boldsymbol{C}_c^{-1}\boldsymbol{C}_r$$

- This equation is combined with the identity $\boldsymbol{v}_r = \boldsymbol{v}_r$:

$$\begin{bmatrix} \boldsymbol{v}_r \\ \boldsymbol{v}_c \end{bmatrix} = \boldsymbol{T}\boldsymbol{v}_r \text{ where } \boldsymbol{T} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{C}_{rc} \end{bmatrix}$$

- The transformation $\boldsymbol{F} = \boldsymbol{T}^T\boldsymbol{F}'$ and $\boldsymbol{K} = \boldsymbol{T}^T\boldsymbol{K}'\boldsymbol{T}$ can be applied to the structural equation $\boldsymbol{F}' = \boldsymbol{K}'\boldsymbol{v}'$

Constraints enforcement – Transformation approach (3)

- Similarly to the condensation procedure presented in Week#2, the structural equations $\boldsymbol{F}' = \boldsymbol{K}'\boldsymbol{v}'$ can be partitioned as

$$\begin{bmatrix} \boldsymbol{K}_{rr} & \boldsymbol{K}_{rc} \\ \boldsymbol{K}_{cr} & \boldsymbol{K}_{cc} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_r \\ \boldsymbol{v}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_r \\ \boldsymbol{F}_c \end{bmatrix}$$

- The condensed system is

$$[\boldsymbol{K}_{rr} + \boldsymbol{K}_{rc}\boldsymbol{C}_{rc} + \boldsymbol{C}_{rc}^T\boldsymbol{K}_{cr} + \boldsymbol{C}_{rc}^T\boldsymbol{K}_{cc}\boldsymbol{C}_{rc}]\boldsymbol{v}_r = [\boldsymbol{F}_r + \boldsymbol{C}_{rc}^T\boldsymbol{F}_c]$$

- After this equation is solved for $\boldsymbol{v}_r$, the displacement corresponding to the condensed out degrees of freedom $\boldsymbol{v}_c$ can be computed using $\boldsymbol{v}_c = \boldsymbol{C}_{rc}\boldsymbol{v}_r$

Example: Beam element with two end springs (1)

- Consider the following structure consisting of a single 2D elastic beam-column element with two inelastic rotational spring at its ends

- Applying the transformation method, write the global structural equation

Example: Beam element with two end springs (2)

- As previously discussed, the following constraints are imposed:

$$\text{Left spring:}\begin{cases} v_1 = v_4 \\ v_2 = v_5 \end{cases} \text{ and right spring:}\begin{cases} v_7 = v_{10} \\ v_8 = v_{11} \end{cases}$$

- The condensed degrees of freedom are selected as v_1, v_2, v_{10} and v_{11}

- The constrained equation are given by

$$[\boldsymbol{C}_r \quad \boldsymbol{C}_c]\begin{bmatrix} \boldsymbol{v}_r \\ \boldsymbol{v}_c \end{bmatrix} = \boldsymbol{0} \rightarrow \left[\begin{array}{cccccccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \begin{bmatrix} v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{12} \\ -- \\ v_1 \\ v_2 \\ v_{10} \\ v_{11} \end{bmatrix} = \boldsymbol{0}$$

Example: Beam element with two end springs (3)

- And

$$\boldsymbol{C}_{rc} = -\boldsymbol{C}_c^{-1}\boldsymbol{C}_r = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

- The transformation matrix is given by

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{C}_{rc} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Example: Beam element with two end springs (4)

- The force-displacement relations are given by
  - For the left spring (spring stiffness $k_{s1} = a_1 EI_e/L$):

$$\begin{bmatrix} F_3 \\ F_6 \end{bmatrix} = \begin{bmatrix} k_{s1} & -k_{s1} \\ -k_{s1} & k_{s1} \end{bmatrix} \begin{bmatrix} v_3 \\ v_6 \end{bmatrix}$$

  - For the right spring (spring stiffness $k_{s2} = a_2 EI_e/L$):

$$\begin{bmatrix} F_9 \\ F_{12} \end{bmatrix} = \begin{bmatrix} k_{s2} & -k_{s2} \\ -k_{s2} & k_{s2} \end{bmatrix} \begin{bmatrix} v_9 \\ v_{12} \end{bmatrix}$$

Example: Beam element with two end springs (5)

- For the elastic beam-column element:

$$\begin{Bmatrix} F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix} = \frac{E}{L}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & aI_e\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & aI_e\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 0 & -aI_e\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & aI_e\frac{6}{L}\left(1+\frac{2}{a_1}\right) \\ 0 & aI_e\frac{6}{L}\left(1+\frac{2}{a_2}\right) & aI_e4\left(1+\frac{3}{a_2}\right) & 0 & -aI_e\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 2aI_e \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & -aI_e\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & -aI_e\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 0 & aI_e\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & -aI_e\frac{6}{L}\left(1+\frac{2}{a_1}\right) \\ 0 & aI_e\frac{6}{L}\left(1+\frac{2}{a_1}\right) & 2aI_e & 0 & -aI_e\frac{6}{L}\left(1+\frac{2}{a_1}\right) & aI_e4\left(1+\frac{3}{a_1}\right) \end{bmatrix}\begin{Bmatrix} v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{Bmatrix}$$

Example: Beam element with two end springs (6)

- The complete system is given by:

$$
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \end{Bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k_{s1} & 0 & 0 & -k_{s1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 0 & -\frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & 0 & 0 & 0 \\
0 & 0 & -k_{s1} & 0 & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & \frac{aEI_e}{L}4\left(1+\frac{3}{a_2}\right)+k_{s1} & 0 & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 2\frac{aEI_e}{L} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 0 & \frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & 2\frac{aEI_e}{L} & 0 & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & \frac{aEI_e}{L}4\left(1+\frac{3}{a_1}\right)+k_{s2} & 0 & 0 & -k_{s2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_{s2} & 0 & 0 & k_{s2}
\end{bmatrix}
\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{Bmatrix}
$$

Example: Beam element with two end springs (7)

- The reduced system is given by

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{v}_r \rightarrow \boldsymbol{T}^T\boldsymbol{F}' = \boldsymbol{T}^T\boldsymbol{K}'\boldsymbol{T}\boldsymbol{v}_r$$

- Which gives

$$\begin{bmatrix} F_3 \\ F_4 + F_1 \\ F_5 + F_2 \\ F_6 \\ F_7 + F_{10} \\ F_8 + F \\ F_9 \\ F_{12} \end{bmatrix} = \begin{bmatrix} k_{s1} & 0 & 0 & -k_{s1} & 0 & 0 & 0 & 0 \\ 0 & \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 \\ 0 & 0 & \frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1 a_2}\right) & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 0 & -\frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1 a_2}\right) & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & 0 \\ -k_{s1} & 0 & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & \frac{aEI_e}{L}4\left(1+\frac{3}{a_2}\right)+k_{s1} & 0 & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 2\frac{aEI_e}{L} & 0 \\ 0 & -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 \\ 0 & 0 & -\frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1 a_2}\right) & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 0 & \frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1 a_2}\right) & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & 0 \\ 0 & 0 & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & 2\frac{aEI_e}{L} & 0 & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & \frac{aEI_e}{L}4\left(1+\frac{3}{a_1}\right)+k_{s2} & -k_{s2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_{s2} & k_{s2} \end{bmatrix} \begin{bmatrix} v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{12} \end{bmatrix}$$

Constraints enforcement – Direct approach (1)

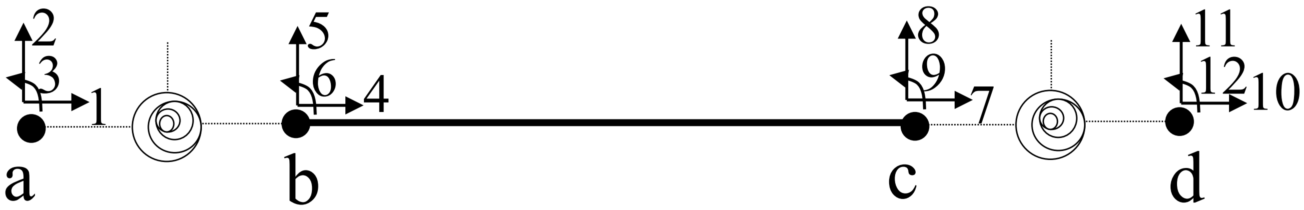
- Another approach to directly enforce constraint which impose **equal degrees of freedom** of the form $v_n = v_m$ (where v_m is to be condensed out), consists in assigning the same number to both degree of freedom number at both nodes.

- The external force acting on degrees of freedom m must be transferred to the degree of freedom n

- It is important to note that this approach is **only applicable to impose equal degrees of freedom constraints** of the form $v_n = v_m$

Constraints enforcement – Direct approach (2)

- For example, consider the previous example:



- As previously discussed, the following constraints are imposed:

Left spring: $\begin{cases} v_1 = v_4 \\ v_2 = v_5 \end{cases}$ and right spring: $\begin{cases} v_7 = v_{10} \\ v_8 = v_{11} \end{cases}$

- The condensed degrees of freedom are selected as v_1, v_2, v_{10} and v_{11}

Constraints enforcement – Direct approach (3)

- The degrees of freedom are transformed as follows:

- With these degrees of freedom, the force-displacement relations are given by
  - For the left spring (spring stiffness $k_{s1} = a_1 EI_e/L$):

$$\begin{bmatrix} F_3 \\ F_6 \end{bmatrix} = \begin{bmatrix} k_{s1} & -k_{s1} \\ -k_{s1} & k_{s1} \end{bmatrix} \begin{bmatrix} v_3 \\ v_6 \end{bmatrix}$$

  - For the right spring (spring stiffness $k_{s2} = a_2 EI_e/L$):

$$\begin{bmatrix} F_9 \\ F_{12} \end{bmatrix} = \begin{bmatrix} k_{s2} & -k_{s2} \\ -k_{s2} & k_{s2} \end{bmatrix} \begin{bmatrix} v_9 \\ v_{12} \end{bmatrix}$$

Constraints enforcement – Direct approach (4)

- For the elastic beam-column element:

$$
\begin{Bmatrix} F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix} = \frac{E}{L}
\begin{bmatrix}
A & 0 & 0 & -A & 0 & 0 \\
0 & aI_e \frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1 a_2}\right) & aI_e \frac{6}{L}\left(1+\frac{2}{a_2}\right) & 0 & -aI_e \frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1 a_2}\right) & aI_e \frac{6}{L}\left(1+\frac{2}{a_1}\right) \\
0 & aI_e \frac{6}{L}\left(1+\frac{2}{a_2}\right) & aI_e 4\left(1+\frac{3}{a_2}\right) & 0 & -aI_e \frac{6}{L}\left(1+\frac{2}{a_2}\right) & 2aI_e \\
-A & 0 & 0 & A & 0 & 0 \\
0 & -aI_e \frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1 a_2}\right) & -aI_e \frac{6}{L}\left(1+\frac{2}{a_2}\right) & 0 & aI_e \frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1 a_2}\right) & -aI_e \frac{6}{L}\left(1+\frac{2}{a_1}\right) \\
0 & aI_e \frac{6}{L}\left(1+\frac{2}{a_1}\right) & 2aI_e & 0 & -aI_e \frac{6}{L}\left(1+\frac{2}{a_1}\right) & aI_e 4\left(1+\frac{3}{a_1}\right)
\end{bmatrix}
\begin{Bmatrix} v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{Bmatrix}
$$

Constraints enforcement – Direct approach (5)

- Assembling the global force-deformation relation directly gives

$$
\begin{bmatrix} F_3 \\ F_4+F_1 \\ F_5+F_2 \\ F_6 \\ F_7+F_{10} \\ F_8+F \\ F_9 \\ F_{12} \end{bmatrix}
=
\begin{bmatrix}
k_{s1} & 0 & 0 & -k_{s1} & 0 & 0 & 0 & 0 \\
0 & \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 \\
0 & 0 & \frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 0 & -\frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & 0 \\
-k_{s1} & 0 & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & \frac{aEI_e}{L}4\left(1+\frac{3}{a_2}\right)+k_{s1} & 0 & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 2\frac{aEI_e}{L} & 0 \\
0 & -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 \\
0 & 0 & -\frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_2}\right) & 0 & \frac{aEI_e}{L}\frac{12}{L^2}\left(1+\frac{a_1+a_2}{a_1a_2}\right) & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & 0 \\
0 & 0 & \frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & 2\frac{aEI_e}{L} & 0 & -\frac{aEI_e}{L}\frac{6}{L}\left(1+\frac{2}{a_1}\right) & \frac{aEI_e}{L}4\left(1+\frac{3}{a_1}\right)+k_{s2} & -k_{s2} \\
0 & 0 & 0 & 0 & 0 & 0 & -k_{s2} & k_{s2}
\end{bmatrix}
\begin{bmatrix} v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{12} \end{bmatrix}
$$

- Which corresponds to the system obtained with the constraint transformation approach