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In-class Exercise Set #10: Numerical integration in beam-column elements

The tapered beam shown in Figure 1 has a linear elastic material. The material modulus of elasticity E is constant. The beam depth changes linearly from $2d$ at the fixed support to d at the tip. The beam width, b, is constant. This beam is analyzed with a single displacement-based beam-column element with two nodes. The left note is *Node-i* and the right node is *Node-j* as shown in the figure.

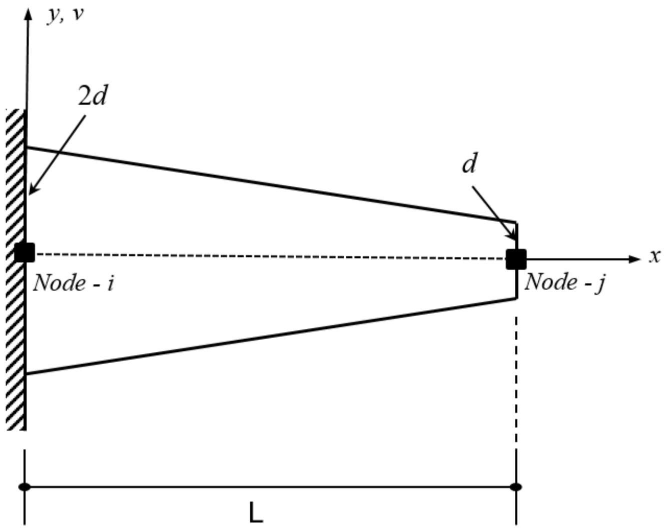

Figure 1. tapered element

The transverse displacement field $v(x)$ along the beam is approximated by the Euler-Bernoulli beam theory assumptions that we discussed in class and,

$$v(x) = \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] v_j + \left[\frac{x^3}{L^2} - \frac{x^2}{L}\right] \theta_j \tag{1}$$

From the above displacement field, the curvature field $k(x)$ can be calculated to get the following transformation matrix,

$$k(x) = \mathbf{B}(\mathbf{x}) \cdot \begin{Bmatrix} v_j \\ \theta_j \end{Bmatrix} \tag{2}$$

With the use of the principle of virtual displacement method, the resulting stiffness matrix of the element is 2x2 and can be calculated as:

$$\mathbf{k} = \int_0^L [\mathbf{B}(\mathbf{x})]^T \mathbf{k}_s(x)\,[\mathbf{B}(\mathbf{x})]dx \tag{3}$$

Where $\mathbf{k}_s(x) = EI(x)$ is the section stiffness matrix.

Answer to the following questions:

1. What type of numerical integration method do you propose in order to calculate the above stiffness matrix "numerically exact"? How many integration points should be used with this method and explain why?
2. Calculate the "numerically exact" stiffness matrix.
3. Is this stiffness matrix the "theoretically exact" stiffness matrix for this tapered beam? Explain your reasoning.

Solution:
Question 1
The Gauss-Legendre quadrature rule is selected as it yields an exact result when integrating all polynomials of degree $2m-1$ or less, where m indicates the number of integration points.

In order to obtain the number m of integration sections along the element length needed to obtain exact results, the order of the polynomial function to be integrated should be determined

The transverse displacement field is governed by the following two shape functions

$$\boldsymbol{N}^T(x) = \begin{bmatrix} N_1(x) \\ N_2(x) \end{bmatrix} = \begin{bmatrix} 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \\ \frac{x^3}{L^2} - \frac{x^2}{L} \end{bmatrix} \tag{4}$$

As shown in Slides 14 and 21 of Lecture 08, the shape functions for the transversal degrees of freedom should be derived twice in order to obtain the matrix $\boldsymbol{B}(x)$

$$\boldsymbol{B}^T(x) = \frac{\partial^2 \boldsymbol{N}^T(x)}{\partial x^2} = \begin{bmatrix} B_1(x) \\ B_2(x) \end{bmatrix} = \begin{bmatrix} \frac{6}{L^2} - \frac{12x}{L^3} \\ \frac{6x}{L^2} - \frac{2}{L} \end{bmatrix} \tag{5}$$

To determine the section stiffness matrix $\mathbf{k}_s(x)$ for the different integration sections along the element length, the expression of the moment of inertia $I(x)$ should be determined.

The beam's depth is given by

$$d(x) = d\left(2 - \frac{x}{L}\right) \tag{6}$$

Hence, the moment of inertia is

$$I(x) = \frac{b \cdot \left(d(x)\right)^3}{12} = \frac{b \cdot d^3}{12}\left(2 - \frac{x}{L}\right)^3 = I_0\left(2 - \frac{x}{L}\right)^3 \tag{7}$$

The element stiffness matrix that is to be computed is

$$\mathbf{k} = \int_0^L [\mathbf{B}(\mathbf{x})]^T \mathbf{k}_s(x)\, [\mathbf{B}(\mathbf{x})] dx = \int_0^L \boldsymbol{g}(x)\, dx \tag{8}$$

With

$$\boldsymbol{g}(x) = E\ \frac{b \cdot d^3}{12}\left(2 - \frac{x}{L}\right)^3 \begin{bmatrix} \frac{6}{L^2} - \frac{12x}{L^3} \\ \frac{6x}{L^2} - \frac{2}{L} \end{bmatrix} \begin{bmatrix} \frac{6}{L^2} - \frac{12x}{L^3}, & \frac{6x}{L^2} - \frac{2}{L} \end{bmatrix} \tag{9}$$

Or in matrix form

$$\boldsymbol{g}(x) = E\,I_0 \begin{bmatrix} \left(2-\frac{x}{L}\right)^3 \left(\frac{6}{L^2}-\frac{12x}{L^3}\right)^2 & \left(2-\frac{x}{L}\right)^3 \left(\frac{6}{L^2}-\frac{12x}{L^3}\right)\left(\frac{6x}{L^2}-\frac{2}{L}\right) \\ \left(2-\frac{x}{L}\right)^3 \left(\frac{6x}{L^2}-\frac{2}{L}\right)\left(\frac{6}{L^2}-\frac{12x}{L^3}\right) & \left(2-\frac{x}{L}\right)^3 \left(\frac{6x}{L^2}-\frac{2}{L}\right)^2 \end{bmatrix} \quad (10)$$

Since $\boldsymbol{g}(x)$ is a fifth order polynomial and the Gauss-Legendre quadrature rule integrates polynomials exactly up to degree $2m-1$, $m=3$ integration points are used along the element length to integrate exactly $\boldsymbol{g}(x)$.

Question 2

As given in Slide 13 of Lecture 09, the following integration point locations r_i and weights w_{ri} expressed in the natural coordinates are used

- $r_1 = -\sqrt{0.6}$, $w_{r1} = \frac{5}{9}$
- $r_2 = 0$, $w_{r2} = \frac{8}{9}$
- $r_3 = \sqrt{0.6}$, $w_{r3} = \frac{5}{9}$

Those values are used to evaluate integrals in the natural domain, that is between -1 and 1. Hence the sum of all the weights is equal to 2 (which is the interval length).
Therefore, in order to evaluate an integral between 0 and L, the following transformations between the natural domain r and the beam domain x should be made

$$x_i = \frac{L}{2} + r_i \cdot \frac{L}{2} \quad (11)$$

And

$$w_{xi} = w_{ri} \cdot \frac{L}{2} \quad (12)$$

This gives

- $x_1 = 0.11L$, $w_{x1} = \frac{5}{18}L$
- $x_2 = \frac{L}{2}$, $w_{x2} = \frac{8}{18}L$
- $x_3 = 0.89L$, $w_{x3} = \frac{5}{18}L$

The function $\boldsymbol{g}(x)$ is now evaluated at the integration points x_i

$$\boldsymbol{g}(x_1) = E\,I_0 \begin{bmatrix} \frac{147.9}{L^4} & -\frac{42.34}{L^3} \\ -\frac{42.34}{L^3} & \frac{12.12}{L^2} \end{bmatrix} \quad (13)$$

$$\boldsymbol{g}(x_2) = E\,I_0 \begin{bmatrix} 0 & 0 \\ 0 & \frac{3.375}{L^2} \end{bmatrix} \quad (14)$$

$$\boldsymbol{g}(x_3) = E\,I_0 \begin{bmatrix} \frac{29.95}{L^4} & -\frac{21.38}{L^3} \\ -\frac{21.38}{L^3} & \frac{15.26}{L^2} \end{bmatrix} \tag{15}$$

Using the selected quadrature rule, the element stiffness matrix is computed

$$\mathbf{k} = \int_0^L \boldsymbol{g}(x)\,dx = \sum_{i=1}^{3} w_{xi} \cdot \boldsymbol{g}(x_i) = E\,I_0 \begin{bmatrix} \frac{49.40}{L^3} & -\frac{17.70}{L^2} \\ -\frac{17.70}{L^2} & \frac{9.11}{L} \end{bmatrix} \tag{16}$$

Question 3
As previously discussed, since 3 Gauss-Legendre integration points have been used to integrate a fifth-degree polynomial, the stiffness matrix computed using the selected quadrature is the "theoretically exact" stiffness matrix.

Note, small round-off errors can be introduced in the numerical integration by finite-precision computations.