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## In-class Exercise – Week #9: Displacement-based elements

The rectangular cross section shown in Figure 1 is made of a material that is described with the constitutive law according to Equation 1. The cross section is subjected to the strain distribution shown in Figure 1.

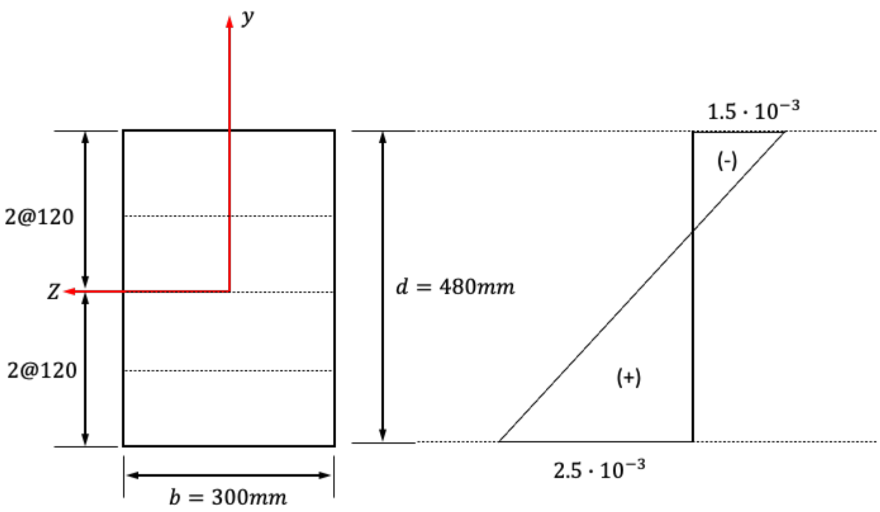


**Figure 1. Cross section and strain distribution**

Stress-strain constitutive law,

$$\sigma(\varepsilon) = signum(\varepsilon) \cdot f_y \cdot \left(1 - e^{-0.5 \cdot \frac{signum(\varepsilon) \cdot \varepsilon}{\varepsilon_y}}\right) \tag{1}$$

Where,

- $signum(\varepsilon)$ provides the sign depending on the sign of the longitudinal strain. However, when inputting compressive strains into Eq. (1), please always use absolute values.
- Yield strength, $f_y = 275MPa$
- Reference modulus, $E = 135GPa$

**Answer the following questions:**

1. Compute the section forces
2. Compute the tangent stiffness matrix of the cross section

**Solution:**

First, the yield strain $\varepsilon_y$ should be computed:

$$\varepsilon_y = \frac{f_y}{E} = 0.002 \tag{1.2}$$

The material tangent modulus corresponds to the derivative of stress with respect to strain is given by:

$$E_t = \frac{d\sigma}{d\varepsilon} = \frac{0.5 f_y}{\varepsilon_y} \cdot e^{-0.5 \cdot \frac{signum(\varepsilon) \cdot \varepsilon}{\varepsilon_y}} \tag{1.3}$$

The given strain distribution consists of an axial strain $\varepsilon_a$ at the neutral axis (y=0) and a curvature $\kappa$.
The total strain $\varepsilon$ is obtained as follows:

$$\varepsilon(y) = \varepsilon_a + y \cdot \kappa \tag{1.4}$$

Using the strain values at the section extremities, the values $\varepsilon_a = 0.5 \cdot 10^{-3}$ and $\kappa = -\frac{1}{120} \cdot 10^{-3}$ are obtained.

**Determine the material state (stress and tangent modulus) at each fiber**
In order to determine the stress and tangent modulus at each fiber, the strain $\varepsilon$ at the fiber centroid should be determined using Equation 1.4. The obtained fiber strain can then be inputted into the material stress-strain constitutive law in order to determine the stress (using Equation 1.1) and the tangent modulus (using Equation 1.3).
The following results are obtained:

| Point | Location | Axial strain $\varepsilon_k = \varepsilon_a + y_k \cdot \kappa$ | Axial stress $\sigma_k = \sigma(\varepsilon_k)$ $[MPa]$ | Tangent Modulus $E_k = E(\varepsilon_k)$ $[GPa]$ |
|---|---|---|---|---|
| $k = 1$ | $y_1 = 180$ | $\varepsilon_1 = -1.0 \cdot 10^{-3}$ | $\sigma_1 = -59.9$ | $E_1 = 52.8$ |
| $k = 2$ | $y_2 = 60$ | $\varepsilon_2 = 0$ | $\sigma_2 = 0$ | $E_2 = 67.5$ |
| $k = 3$ | $y_3 = -60$ | $\varepsilon_3 = 1.0 \cdot 10^{-3}$ | $\sigma_3 = 59.9$ | $E_3 = 52.8$ |
| $k = 4$ | $y_4 = -180$ | $\varepsilon_4 = 2.0 \cdot 10^{-3}$ | $\sigma_4 = 106.7$ | $E_4 = 40.9$ |

**Determination of section forces and tangent stiffness by integration**

The fiber stresses and tangent modulus are integrated throughout the cross-section to determine the section forces $\boldsymbol{s}$ and tangent stiffness $\boldsymbol{k_s}$ :

$$\boldsymbol{s} = \int_A \begin{pmatrix} 1 \\ -y \end{pmatrix} \sigma(y) dA \tag{1.5}$$

$$\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{e}} = \boldsymbol{k_s}(x) = \int_A \frac{d\sigma(y)}{d\varepsilon} \begin{bmatrix} 1 & -y \\ -y & y^2 \end{bmatrix} dA \tag{1.6}$$

The integrals can be evaluated for different fibers by using the midpoint integration rule:

$$\boldsymbol{s} = \sum_k \begin{pmatrix} 1 \\ -y_k \end{pmatrix} \sigma_k A_k \quad (1.7)$$

$$\boldsymbol{k_s}(x) = \sum_k E_k \begin{bmatrix} 1 & -y_k \\ -y_k & y_k^2 \end{bmatrix} A_k \quad (1.8)$$

Where $E_k$ is the tangent material modulus and $A_k$ is the fiber area.
Equations 1.7 and 1.8 are evaluated using the proposed fiber discretization as follows:

$$\boldsymbol{s} = \sum_{k=1}^{4} \begin{pmatrix} 1 \\ -y_k \end{pmatrix} \sigma_k A_k = \begin{pmatrix} 3.84 \\ 1208 \end{pmatrix} 10^6 \quad (1.9)$$

$$\boldsymbol{k_s}(x) = \sum_k E_k \begin{bmatrix} 1 & -y_k \\ -y_k & y_k^2 \end{bmatrix} A_k = \begin{bmatrix} 7.70 & -109.2 \\ -109.2 & 124910 \end{bmatrix} 10^9 \quad (1.10)$$