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In-class Exercise – Week #8: Zero length elements for material nonlinearity

Exercise #1:

Derive the stiffness matrix for an element that is comprised of a zero-length rotational element and an elastic beam-column element as shown in Figure 1.1 in the basic reference system. Assume that the rotational stiffness of the zero-length element is $n \frac{3EI_e}{L}$ where $I_e = \frac{n+1}{n} I$.

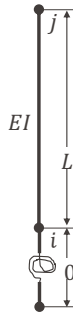


Figure 1.1. Zero length rotational element and elastic beam-column element in series

Hint: By using static condensation, determine the coefficients S_{22} , S_{23} , S_{32} and S_{33} of the stiffness matrix $\hat{\mathbf{k}}_{mod}$ of the elastic beam-column element given by

$$\hat{\mathbf{k}}_{mod} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{S_{22}EI_e}{L} & \frac{S_{23}EI_e}{L} \\ 0 & \frac{S_{32}EI_e}{L} & \frac{S_{33}EI_e}{L} \end{bmatrix}$$

such that a beam-column member can be modeled with the derived elastic beam-column element and a rotational spring.

Recall that the elastic stiffness matrix, $\hat{\mathbf{k}}$ of an elastic beam-column element in the basic reference system is as follows:

$$\hat{\mathbf{k}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{4EI}{L} & \frac{2EI}{L} \\ 0 & \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$$

Solution:

Consider the following beam-column element with a spring at node 1

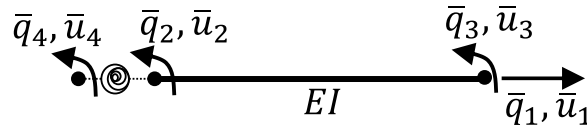


Figure 1.2. Elastic beam-column element with one spring in the basic reference frame

The following moment-rotation relation can be written for the rotational spring:

$$\begin{pmatrix} \bar{q}_2 \\ \bar{q}_4 \end{pmatrix} = k_s \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} \bar{u}_2 \\ \bar{u}_4 \end{pmatrix} \quad (1.1)$$

The following force-displacement relation can be written for the elastic beam-column element:

$$\begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \end{pmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{S_{22}EI_e}{L} & \frac{S_{23}EI_e}{L} \\ 0 & \frac{S_{32}EI_e}{L} & \frac{S_{33}EI_e}{L} \end{bmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{pmatrix} \quad (1.2)$$

Combining Equations 1.3 and 1.4 gives

$$\begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \\ \bar{q}_4 \end{pmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 \\ 0 & k_s + \frac{S_{22}EI_e}{L} & \frac{S_{23}EI_e}{L} & -k_s \\ 0 & \frac{S_{32}EI_e}{L} & \frac{S_{33}EI_e}{L} & 0 \\ 0 & -k_s & 0 & k_s \end{bmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{pmatrix} \quad (1.3)$$

The objective is now to eliminate the degree of freedom corresponding to $\bar{u}_2$ by using the static condensation method.

Using Slides 29 to 31 from Lecture #1, the problems can be decomposed as follows:

$$\mathbf{K}_{bb} = \left[k_s + \frac{S_{22}EI_e}{L} \right] \quad (1.4)$$

$$\mathbf{K}_{bc} = \begin{bmatrix} 0 & \frac{S_{23}EI_e}{L} & -k_s \end{bmatrix} \quad (1.5)$$

$$\mathbf{K}_{cb} = \begin{bmatrix} 0 & \frac{S_{32}EI_e}{L} & -k_s \end{bmatrix}^T \quad (1.6)$$

$$\mathbf{K}_{cc} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{S_{33}EI_e}{L} & 0 \\ 0 & 0 & k_s \end{bmatrix} \quad (1.7)$$

Therefore, the following condensed stiffness matrix is obtained

$$\begin{aligned} \hat{\mathbf{K}}_{cc} &= \mathbf{K}_{cc} - \mathbf{K}_{cb}(\mathbf{K}_{bb})^{-1}\mathbf{K}_{bc} \\ &= \begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ 0 & \frac{EI_e S_{33}}{L} - \frac{E^2 I_e^2 S_{23} S_{32}}{L^2 \left(k_s + \frac{EI_e S_{22}}{L} \right)} & \frac{EI_e S_{32} k_s}{L k_s + EI_e S_{22}} \\ 0 & \frac{EI_e S_{23} k_s}{L k_s + EI_e S_{22}} & \frac{EI_e S_{22} k_s}{L k_s + EI_e S_{22}} \end{bmatrix} \end{aligned} \quad (1.8)$$

Substituting the definitions of k_s and I_e given by

$$k_s = n \frac{3EI_e}{L} \text{ and } I_e = \frac{n+1}{n} I$$

into Equation 1.8 gives

$$\hat{\mathbf{K}}_{cc} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ 0 & \frac{EI(n+1)(S_{22}S_{33} - S_{23}S_{32} + 3S_{33}n)}{Ln(S_{22} + 3n)} & \frac{3EIS_{32}(n+1)}{L(S_{22} + 3n)} \\ 0 & \frac{3EIS_{23}(n+1)}{L(S_{22} + 3n)} & \frac{3EIS_{22}(n+1)}{L(S_{22} + 3n)} \end{bmatrix} \quad (1.9)$$

This matrix should be equal to the stiffness matrix of the member in the basic reference frame $\hat{\mathbf{k}}$. Therefore, the following system of equations is obtained

$$\left\{ \begin{aligned} \frac{EI(n+1)(S_{22}S_{33} - S_{23}S_{32} + 3S_{33}n)}{Ln(S_{22} + 3n)} &= \frac{4EI}{L} \\ \frac{3EIS_{32}(n+1)}{L(S_{22} + 3n)} &= \frac{2EI}{L} \\ \frac{3EIS_{23}(n+1)}{L(S_{22} + 3n)} &= \frac{2EI}{L} \\ \frac{3EIS_{22}(n+1)}{L(S_{22} + 3n)} &= \frac{4EI}{L} \end{aligned} \right. \quad (1.10)$$

Solving this system of equations finally gives

$$S_{22} = \frac{12n}{3n - 1} \quad (1.11)$$

$$S_{23} = S_{32} = \frac{6n}{3n - 1} \quad (1.12)$$

$$S_{33} = \frac{12n^2}{3n^2 + 2n - 1} \quad (1.13)$$

The stiffness matrix of the modified elastic beam-column element is therefore given by

$$\hat{\mathbf{k}}_{mod} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{S_{22}EI_e}{L} & \frac{S_{23}EI_e}{L} \\ 0 & \frac{S_{32}EI_e}{L} & \frac{S_{33}EI_e}{L} \end{bmatrix} \quad (1.14)$$

Exercise #2:

Write a script for a zero-length rotational element that utilizes the moment-rotation constitutive relation shown in Figure 2.1. Consider both the loading and unload paths in your model. Validate your implementation for the following input parameters:

- $k_e = 400000 \text{ kN.mm/rad}$
 - $\theta_p = \theta_c - \theta_y = 0,02 \text{ rad}$
 - $\theta_{pc} = \theta_u - \theta_c = 0,05 \text{ rad}$
 - $M_y^* = 4000 \text{ kN.mm}$ and $M_u = 4500 \text{ kN.mm}$
1. Load case #1: Rotational monotonic loading, $\theta = \{0, 0,08\}^T \text{ rad}$
 2. Load case #2: Rotational cyclic loading, $\theta = \{0, 0,06, -0,06, 0\}^T \text{ rad}$

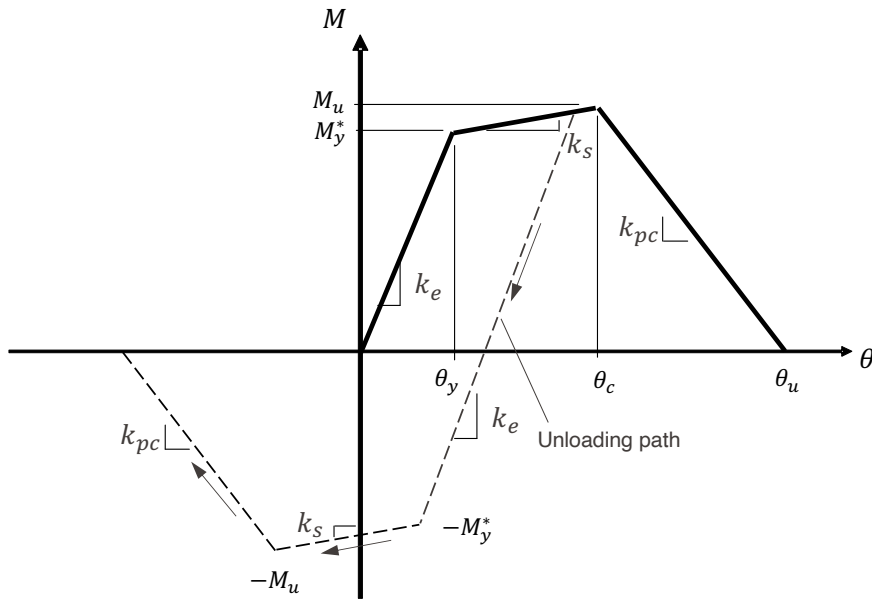


Figure 2.1. Moment-rotation relation for zero length element

Solution:

The following steps summarize the workflow of the implemented constitutive formulation at a load step n :

- 1) Determine if the loading is moving towards positive or negative rotation: if $\theta^i > \theta^{i-1}$, the loading is moving towards positive rotation
- 2) Compute the trial elastic moment based on the previous moment and rotation using the elastic tangent stiffness k_e :

$$M^i = M^{i-1} + k_e(\theta^i - \theta^{i-1})$$

- 3) Check if the material is in the residual path (i.e. if $M^i = 0$ and $k^i = 0$)
- 4) If the material is an inelastic path and moving towards positive rotation (i.e. if $M^i > M_{max}^+$):
 - 4.1) Reset the negative yielding and capping flags
 - 4.2) Update the initial rotation offset in the positive direction
 - 4.3) If the spring is the hardening path or softening path, compute the moment accordingly
 - 4.4) If rotation exceeds the θ_u , the residual flag is set, and the moment is set to zero, indicating failure
 - 4.5) Update the maximum moment which corresponds to the positive backbone curve
- 5) If the material is an inelastic path and moving towards negative rotation (i.e. if $M^i < M_{max}^-$): analogous to steps 4.1 to 4.5 above
- 6) Compute the tangent stiffness k :

$$k = \frac{M^i - M^{i-1}}{\theta^i - \theta^{i-1}}$$

The following responses are obtained for the monotonic and cyclic load protocols:

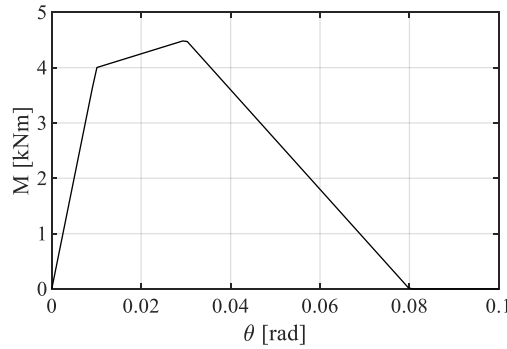


Figure 2.2. Spring response under monotonic load

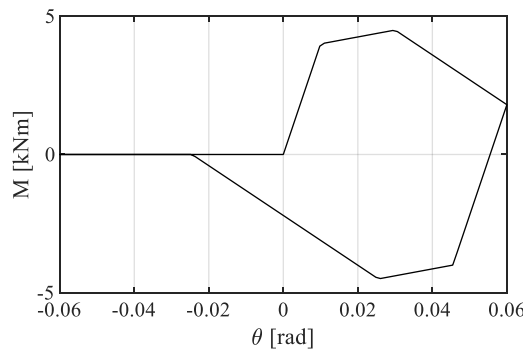


Figure 2.3. Spring response under cyclic load

Exercise #3:

Consider the following column from Week #5:

$$A = 1.27 \cdot 10^4 \text{ mm}^2, I = 3.66 \cdot 10^7 \text{ mm}^4, E = 200,000 \text{ MPa}$$

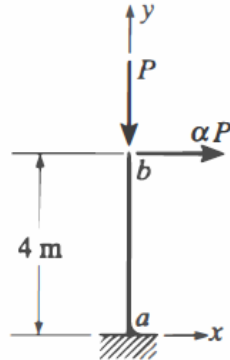


Figure 3.1. Column under axial and lateral loading

Consider material nonlinearity with the element model you developed in Exercise #2 and the element stiffness matrix you derived in Exercise #1. For $\alpha=0.05$, determine the load-displacement relationship (secondary equilibrium path) of the cantilever member when:

- $\theta_p = \theta_c - \theta_y = 0.02 \text{ rad}$
 - $\theta_{pc} = \theta_u - \theta_c = 0.05 \text{ rad}$
 - $M_y^* = 4000 \text{ kN.m}$ and $M_u = 4500 \text{ kN.m}$
1. Would you use a displacement or load-control scheme for your solution? Explain your answer.
 2. Compare the computed secondary equilibrium path for the following cases:
 - a. Case #1: Linear elastic analysis (from Week #5)
 - b. Case #2: Nonlinear geometric analysis and linear material (from Week #5)
 - c. Case #3: Nonlinear analysis with material nonlinearity and linear geometric transformation
 - d. Case #4: Nonlinear analysis with both material and geometric nonlinearities
 3. Calculate the displacement at which the cantilever member reaches zero lateral strength (i.e., collapse) with your program.

Solution:

- 1) Displacement-control is used because the inelastic spring exhibits a snap-trough behavior.
- 2) To determine the load-displacement relation of the member when it is modeled using an elastic beam-column element and an inelastic spring, the same steps are used as shown in the solution of Exercise 1 of Week 5, with the following modifications when computing the structure resisting force vector and the tangent stiffness matrix.

The following global degrees of freedom are used:

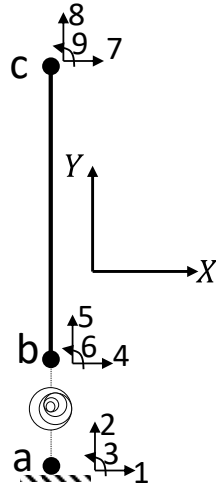


Figure 3.2. Global degrees of freedom

Referring to Figure 3.2, the global degrees of freedom 1 and 2 should be equal to 4 and 5, respectively. This constraint can be enforced by removing the global degrees of freedom 1 and 2 and applying the fixed boundary conditions to the global degrees of freedom 4 and 5. With this approach, the following global degrees of freedom are obtained:

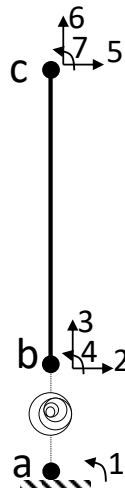


Figure 3.3. Global degrees of freedom with equal DOF constraint

The global degrees of freedom corresponding to the inelastic spring are therefore 1 and 4, while those of the elastic beam-column elements are 2, 3, 4, 5, 6 and 6.

At each iteration i of the iterative procedure for the displacement-control algorithm, the element state determination for the elastic beam-column element is done similarly to the one described in the solution of Exercise 1 of Week 5. The only difference is that the element stiffness matrix derived in Question 1 is used.

For the inelastic spring element, the following state determination is used:

- The spring rotation θ is defined as $\theta = v_4 - v_1$. Note that this convention become very important in the case where the constitutive relation assigned to the spring exhibits an asymmetric response between tension and compression. This rotation is then inputted into the constitutive formulation derived in Question 2, which return the spring moment M_{spring} and tangent stiffness k_{spring} , which are used to form the spring resisting force vector $\mathbf{F}_{int,spring}$ and stiffness matrix $\mathbf{K}_{spring}$ as follows:

$$\mathbf{F}_{int,spring} = \begin{pmatrix} -M_{spring} \\ M_{spring} \end{pmatrix}$$

$$\mathbf{K}_{spring} = k_{spring} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

These quantities are then used when assembling the structure resisting force vector and stiffness matrix.

It is also important to note that to compute the constitutive relation for the inelastic spring formulated in Question 2, the previous moment and previous rotation are needed. These refer to the previous converged step and should only be updated for each displacement step once the iterative procedure for the displacement-control algorithm has converged.

The following figure compares the results obtained for the case where $M_y^* = 4000kN.mm$ and $M_u = 4500kN.mm$.

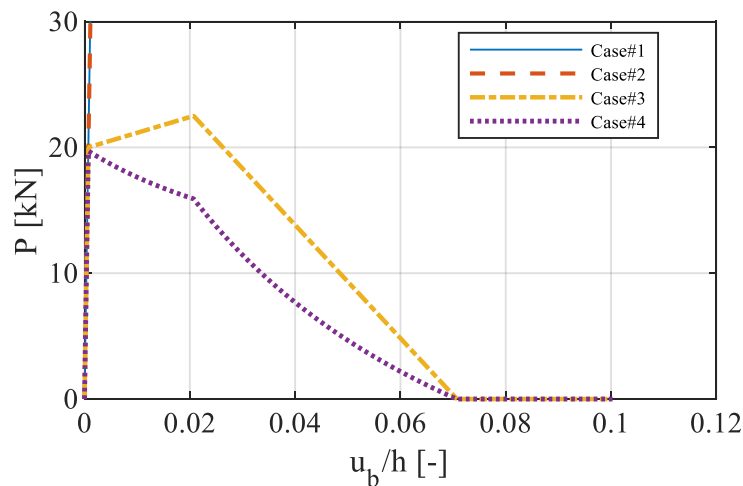


Figure 3.4. Comparison of computed secondary equilibrium paths under various analysis methods

Similarly, the figure below compares the results obtained for a second case where $M_y^* = 200000kN.mm$ and $M_u = 225000kN.mm$.

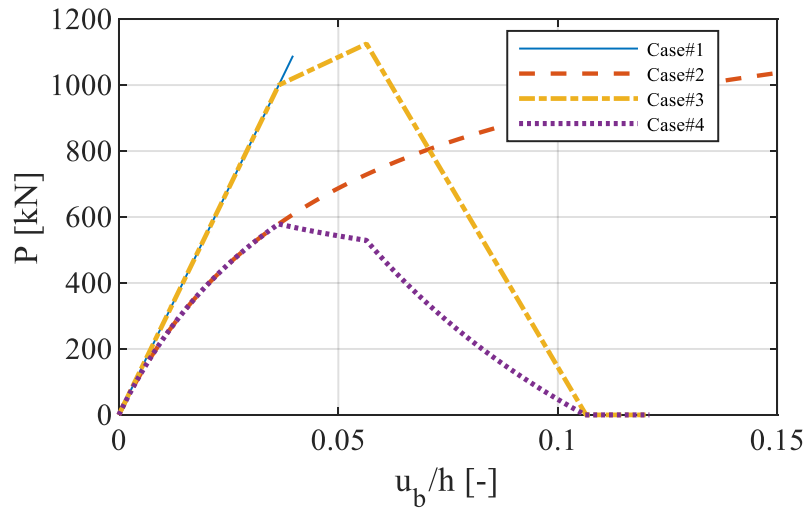


Figure 3.5. Comparison of computed secondary equilibrium paths under various analysis methods – increased yield and capping moments