

CIVIL 449: Nonlinear Analysis of Structures

School of Architecture, Civil & Environmental Engineering
Civil Engineering Institute

Geometric Nonlinear Analysis

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EPFL Objectives of today's lecture

- To introduce:
 - Geometric stiffness matrix
 - Basic reference system for frame elements
 - Element transformations (from local to basic coordinate system)
 - Corotational transformation

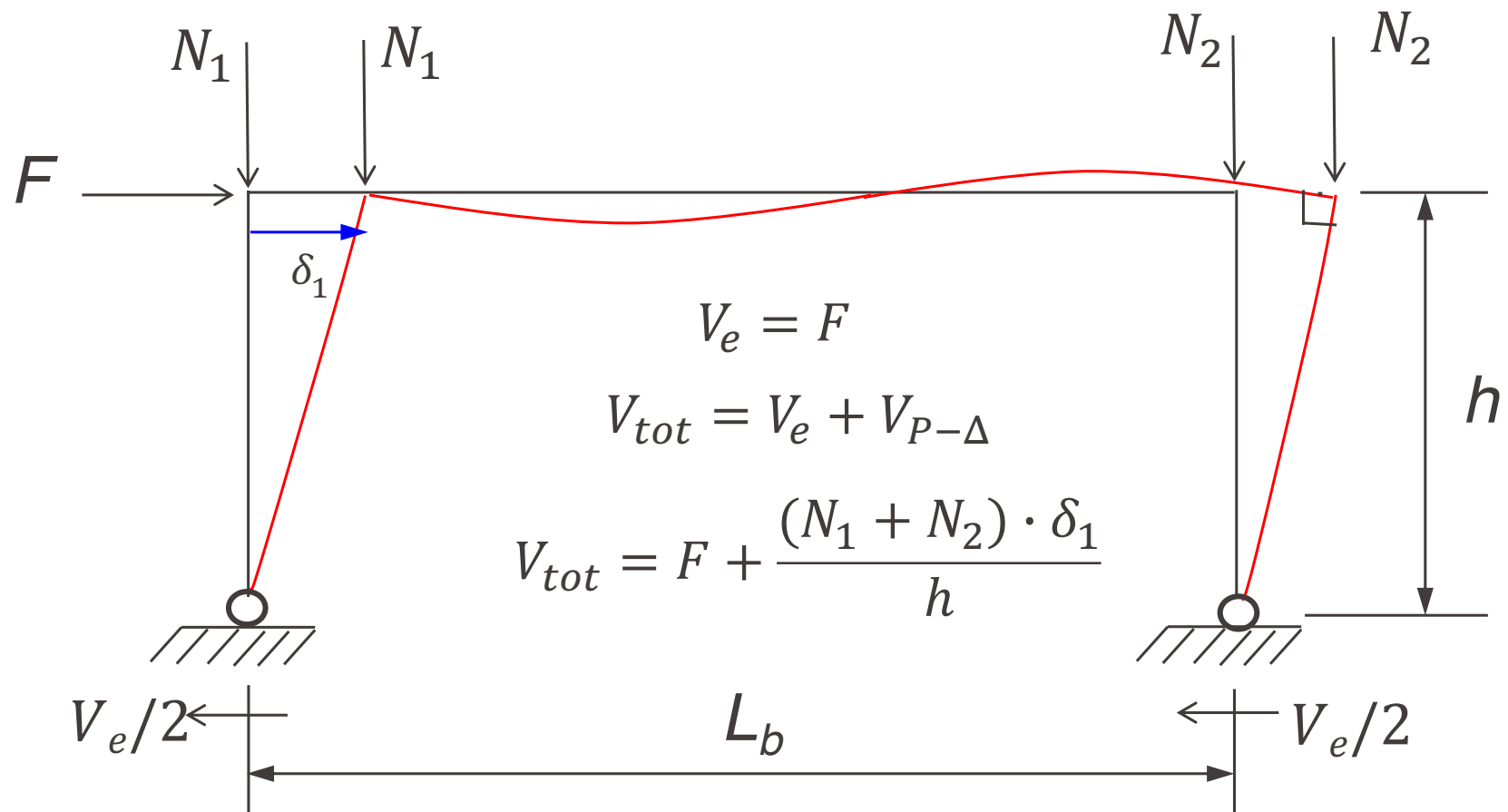
EPFL Deflections

- Strength and stiffness are completely different properties of a member, that are indeed related to each other.
- A fishing rod is flexible yet strong.
- Floor systems and structures cannot deflect as much for several reasons.

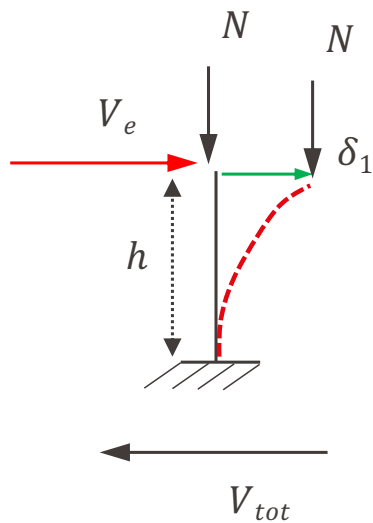


Source: <http://www.africancichlid.com/Structure.htm>

Deflections cause second order effects

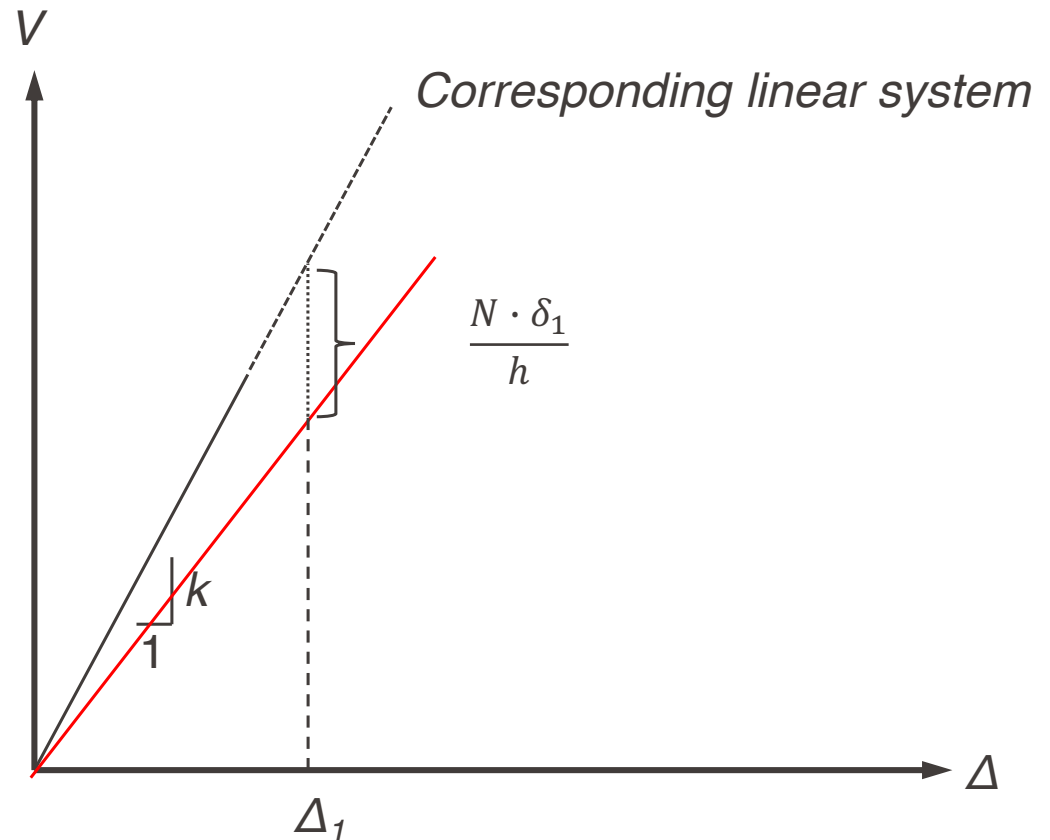


Deflections cause second order effects (2)

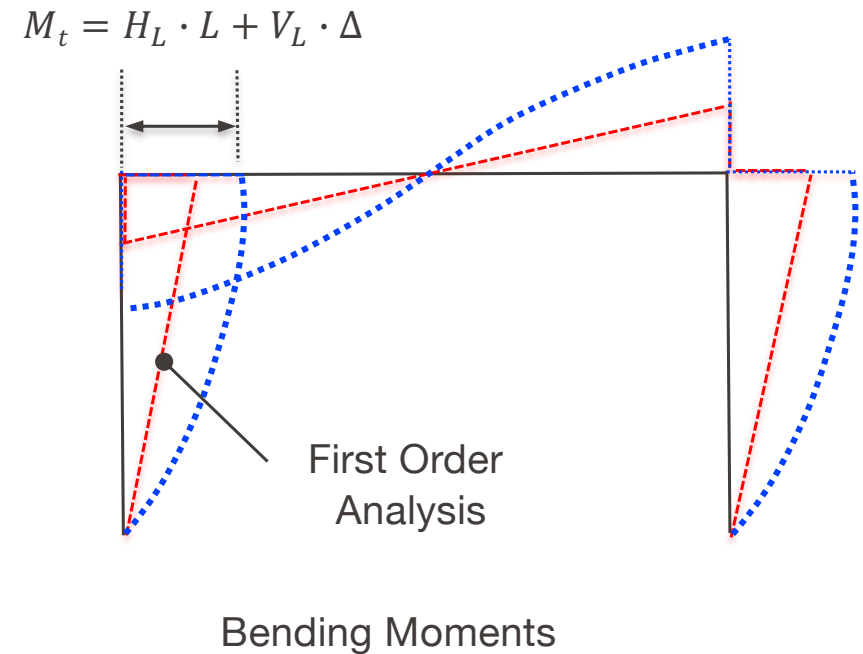
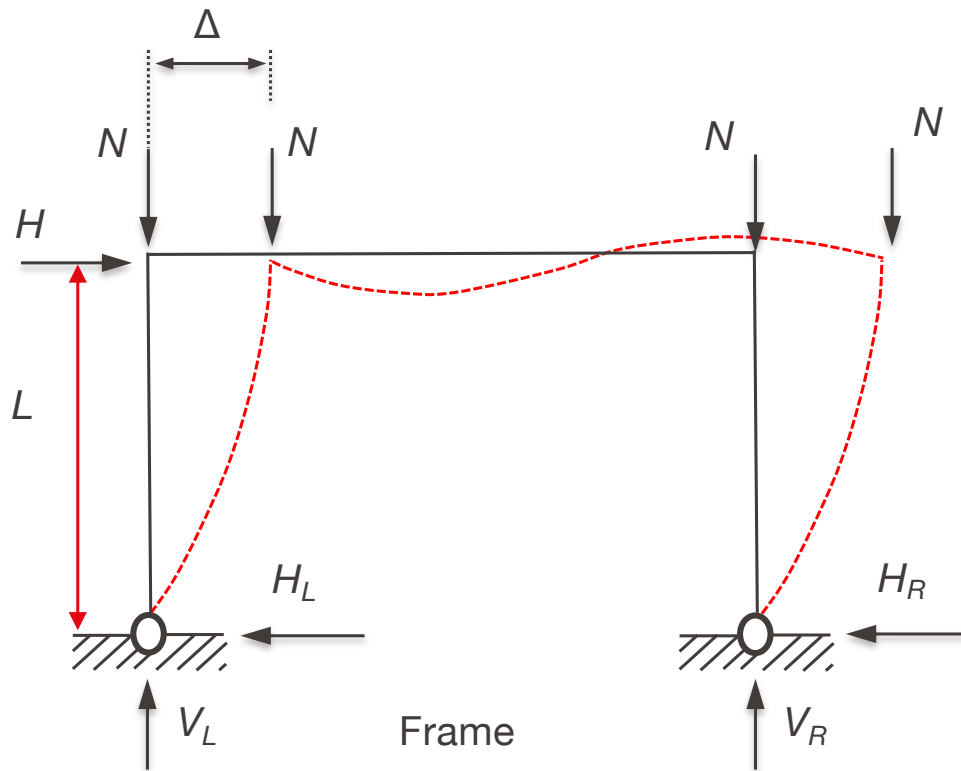


$$V_{tot} = V_e + V_{P-\Delta}$$

$$V_{tot} = V_e + \frac{N \cdot \delta_1}{h}$$



EPFL P-Delta effects on frame structures



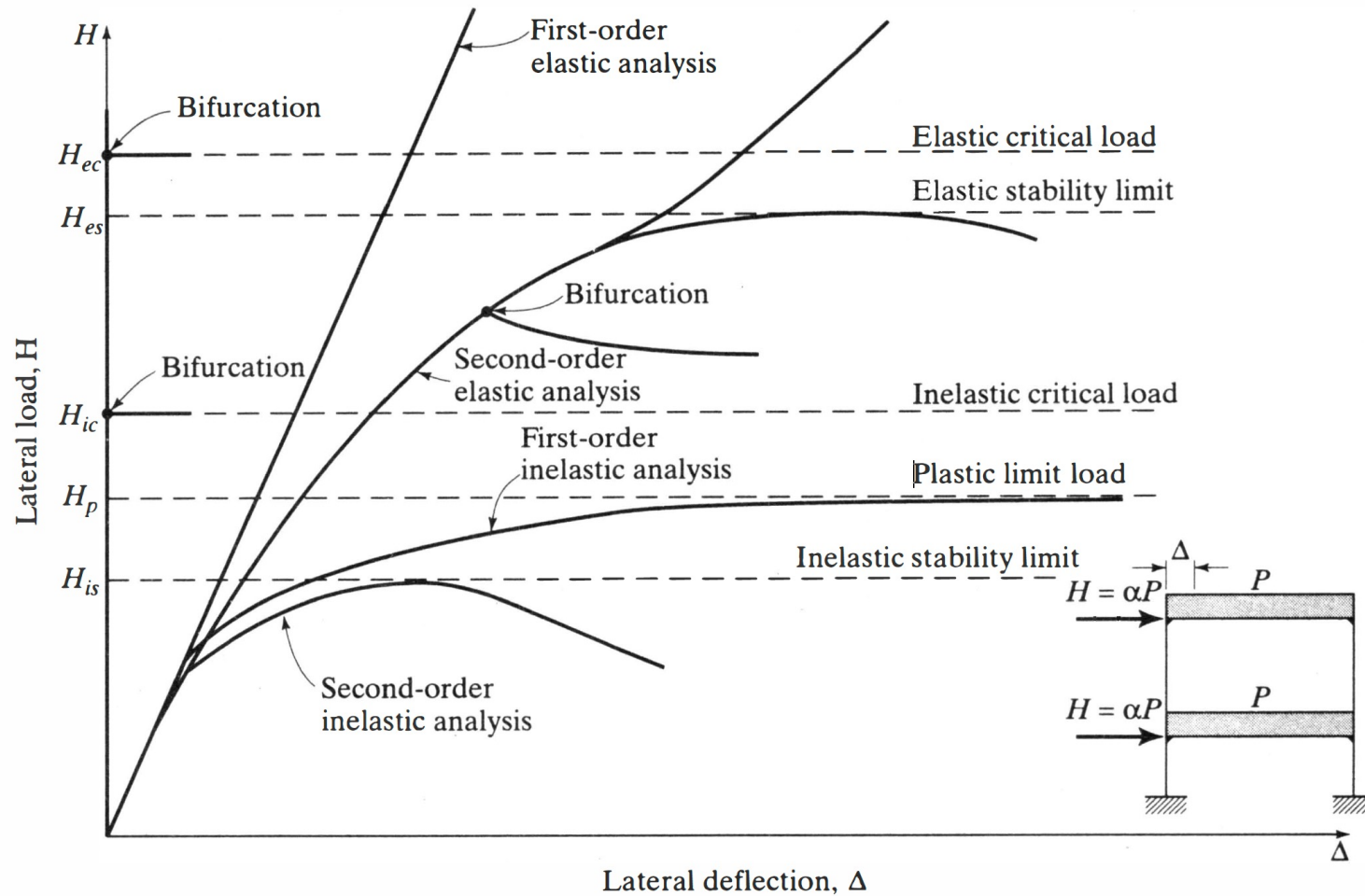
- $N \cdot \Delta$: Additional moment (couple) due to the axial force acting through the relative transverse displacement of member ends

EPFL Types of analysis

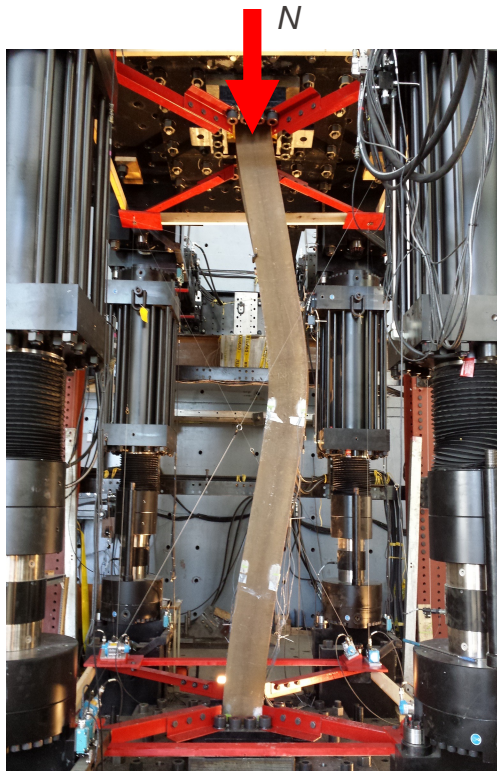
1. **First Order Elastic Analysis:** The equations of equilibrium are always written in the undeformed configuration and material nonlinearity is not considered.
2. **Second Order Elastic Analysis:** The equations of equilibrium are always written in the deformed configuration and material nonlinearity is not considered.
3. **First Order inelastic Analysis:** The equations of equilibrium are always written in the undeformed configuration and material nonlinearity is considered.
4. **Second Order Inelastic Analysis:** The equations of equilibrium are always written in the deformed configuration and material nonlinearity is considered.

Normally they require special software

EPFL Types of analysis (2)



EPFL Several examples from established theory



@Prof. R. Tremblay (2015)



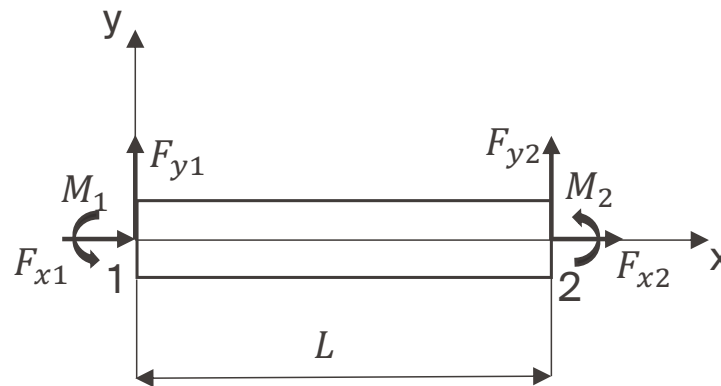
Lignos et al. (2013)



@Prof. E. Miranda (2017)

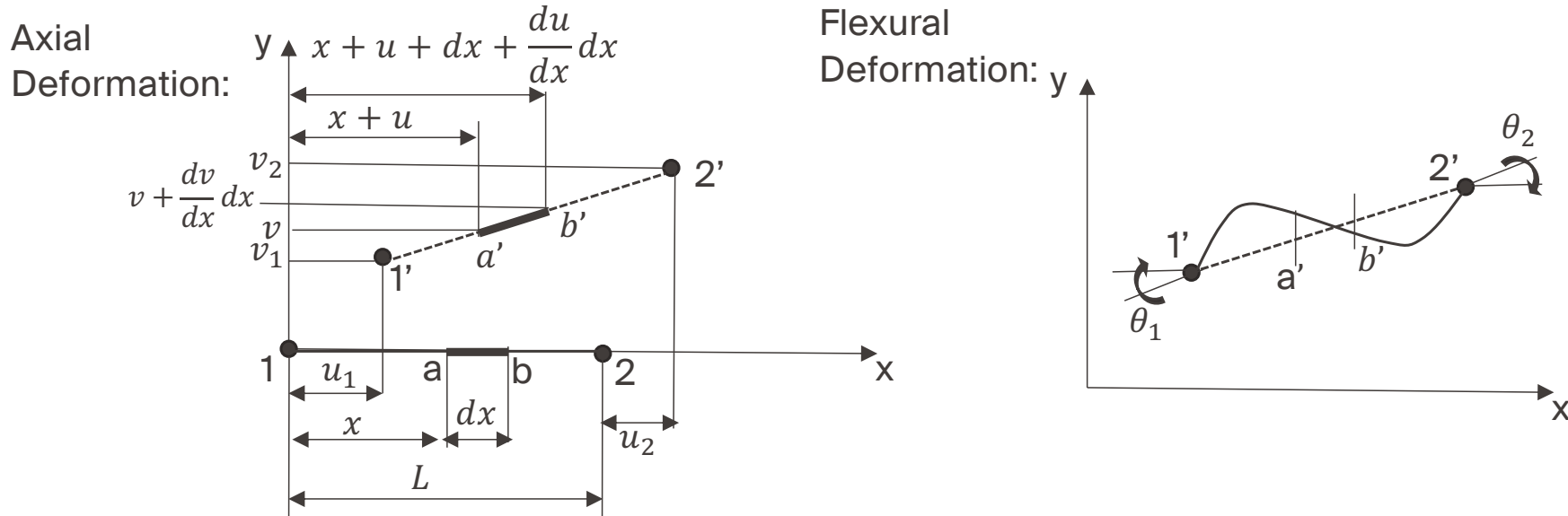
Geometric stiffness matrix for planar elements

- Instead of using infinitesimal strains that form the basis of linear analysis, start using small finite strains
- Combination of bending and axial force
- Consider the 2d elastic beam element that we saw previously:



Geometric stiffness matrix for planar elements (2)

- Consider the effect of both axial and flexural deformations



- Consider only the axial deformation; denote by ab the length of the infinitesimal segment dx (i.e., $ab = dx$) in the reference configuration
- After rigid body rotation and axial deformation, the length of the segment dx is as follows:

$$a'b' = \left[\left(dx + \frac{du}{dx} dx \right)^2 + \left(\frac{dv}{dx} dx \right)^2 \right]^{\frac{1}{2}} = \left[1 + 2 \frac{du}{dx} + \left(\frac{du}{dx} \right)^2 + \left(\frac{dv}{dx} \right)^2 \right]^{\frac{1}{2}} dx$$

EPFL Geometric stiffness matrix for planar elements (3)

- Using the binomial theorem and neglecting the higher-order terms gives

$$\frac{a'b'}{dx} = 1 + \frac{du}{dx} + \frac{1}{2} \left[\left(\frac{du}{dx} \right)^2 + \left(\frac{dv}{dx} \right)^2 \right]$$

- The finite strain e_{fin} is defined as the sum of the extension per unit length (for the axial deformation) and the strain from the flexural deformation:

$$e_{fin} = \frac{a'b' - ab}{dx} - y \left(\frac{d^2v}{dx^2} \right) = \frac{du}{dx} + \frac{1}{2} \left[\left(\frac{du}{dx} \right)^2 + \left(\frac{dv}{dx} \right)^2 \right] - y \left(\frac{d^2v}{dx^2} \right)$$

Where the last term ($\frac{d^2v}{dx^2}$:= curvature) is the infinitesimal flexural strain (i.e., neglect the effects of the rotation and stretching of the element neutral axis)

- The theorem of virtual displacements (for a uniaxial stress state) is applied to the reference configuration:

$$\delta W_{int} = \int_V \sigma_x \delta e_{fin} dV$$

- The following relations are useful for the next step:

$$\sigma_x = -\frac{yM}{I} \text{ with } I = \int_A y^2 dA$$

$$\delta \left(\frac{du}{dx} \right) = \frac{d\delta u}{dx} \text{ and } \delta \left(\frac{dv}{dx} \right) = \frac{d\delta v}{dx} \text{ which are valid for infinitesimal displacements}$$

Geometric stiffness matrix for planar elements (4)

- Using the definition of e_{fin} , the previous quantities and integrating over the section depth gives

$$\delta W_{int} = \int_0^L \sigma_x A \left(\frac{d\delta u}{dx} \right) dx + \int_0^L M \left(\frac{d^2 v}{dx^2} \right) dx + \frac{1}{2} \int_0^L \sigma_x A \left[\delta \left(\frac{du}{dx} \right)^2 + \delta \left(\frac{dv}{dx} \right)^2 \right] dx$$

- The following relations are useful for the next step:

$$\sigma_x = \left(\frac{du}{dx} \right) E, F_{x2} = \sigma_x A, M = \left(\frac{d^2 v}{dx^2} \right) EI$$

- The virtual work can be rewritten as

$$\delta W_{int} = \int_0^L \left(\frac{du}{dx} \right) EA \left(\frac{d\delta u}{dx} \right) dx + \int_0^L \left(\frac{d^2 v}{dx^2} \right) EI \left(\frac{d^2 \delta v}{dx^2} \right) dx + \frac{1}{2} F_{x2} \int_0^L \left[\delta \left(\frac{du}{dx} \right)^2 + \delta \left(\frac{dv}{dx} \right)^2 \right] dx$$

- The elastic stiffness matrix $\mathbf{k}_e$ (both for axial and flexural deformation) follows from the first two integrals
- The third integral produces a geometric stiffness matrix $\mathbf{k}_g$
- To compute the third integral, the “mathematical trick” is that the virtual operator δ may be treated as a differential operator with respect to the variables $\frac{du}{dx}$ and $\frac{dv}{dx} \rightarrow$ for variable u : $\delta \left(\frac{du}{dx} \right)^2 = 2 \frac{d\delta u}{dx} \frac{du}{dx}$

Geometric stiffness matrix for planar elements (5)

- The third term of the virtual work can then be rewritten as

$$\delta W_{int,3} = F_{x2} \int_0^L \left[\left(\frac{d\delta u}{dx} \frac{du}{dx} \right) + \left(\frac{d\delta v}{dx} \frac{dv}{dx} \right) \right] dx$$

- The displacements are interpolated using the usual shape functions $u\left(\frac{x}{L}\right) = \mathbf{N}\mathbf{u}$:
 - For an axial member:

$$\mathbf{u} = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

- For a flexural member (see lecture notes from last week):

$$v = \begin{bmatrix} 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 & x - 2x\left(\frac{x}{L}\right) + x\left(\frac{x}{L}\right)^2 & 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 & \frac{x^2}{L}\left(-1 + \frac{x}{L}\right) \end{bmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

- The derivative of the displacement with respect to x are given by

$$\frac{du}{dx} = \frac{d\mathbf{N}}{dx} \mathbf{u} = \mathbf{N}' \mathbf{u}$$

- Similarly,

$$\delta u = \mathbf{N} \delta \mathbf{u} \text{ and } \frac{d\delta u}{dx} = \mathbf{N}' \delta \mathbf{u}$$

Geometric stiffness matrix for planar elements (6)

- The third term of the virtual work can then be rewritten as

$$\delta W_{int,3} = \delta \mathbf{u}^T \left(F_{x2} \int_0^L [\mathbf{N}_u^T \mathbf{N}_u + \mathbf{N}_v^T \mathbf{N}_v] dx \right) \mathbf{u}$$

Where $\mathbf{N}_u$ and $\mathbf{N}_v$ are the row vectors of the shape functions for the axial and flexural member, respectively.

NOTE: The (+) sign shall be interpreted as an assembly procedure for the corresponding degrees of freedom.

- Recognizing that from the virtual work theorem, the internal work should be equal to the external work $\delta W_{ext} = \delta \mathbf{u}^T \mathbf{F}$ and using $\mathbf{F} = \mathbf{k} \mathbf{u}$, the following local geometric stiffness matrix is obtained:

$$\mathbf{k}_g^{local} = F_{x2} \int_0^L [\mathbf{N}_u^T \mathbf{N}_u + \mathbf{N}_v^T \mathbf{N}_v] dx$$

Geometric stiffness matrix for planar elements (7)

- And after multiplying and integrating

$$\mathbf{k}_g^{local} = \frac{F_{x2}}{L} \begin{matrix} & \begin{matrix} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2L^2}{15} & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & \frac{2L^2}{15} \end{bmatrix} & \begin{matrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{matrix} \end{matrix}$$

- To assemble the local geometric stiffness matrices to the global geometric stiffness matrix, the usual assembly procedure presented in the previous lectures may be used.

Elastic stiffness matrix for planar elements

- Recall, the elastic stiffness matrix $\mathbf{k}_e^{local}$ for elastic 2d beam elements (presented in the previous lecture) is given as a reminder:

$$\mathbf{k}_e^{local} = \frac{EI}{L^3} \begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix}$$

EPFL Geometric stiffness matrix

- The geometric stiffness matrix can be used to compute the elastic critical loads which will lead to flexural buckling and/or lateral torsional buckling. The predicted instability mode depends on the degrees of freedom present in the element formulation.
- To compute the elastic critical load, the global stiffness equation is written in the form of a generalized eigenvalue problem; the equation of equilibrium at the critical state is:

$$[\mathbf{K}_e^{global} + \lambda \hat{\mathbf{K}}_g^{global}] \Delta = \mathbf{0}$$

where $\mathbf{K}_e^{global}$ is the global elastic stiffness matrix, $\hat{\mathbf{K}}_g^{global}$ is the global geometric stiffness matrix computed for a reference load, $\mathbf{P}_{ref}$; λ (an eigenvalue) is the load factor with respect to $\mathbf{P}_{ref}$ and Δ (an eigenvector) is the buckled shape.

- The lowest value of λ that satisfies the equation for $\Delta \neq \mathbf{0}$ gives the elastic critical load vector $\lambda \mathbf{P}_{ref}$ and the corresponding, Δ defines the buckled configuration.

EPFL Solution of the eigenvalue problem

- To determine the eigenvalues and eigenvector, the eigenvalue problem is rewritten as

$$\mathbf{K}_{ef}^{global} \Delta_f = -\lambda \hat{\mathbf{K}}_{gf}^{global} \Delta_f \Leftrightarrow \frac{1}{\lambda} \mathbf{K}_{ef}^{global} \Delta_f = -\hat{\mathbf{K}}_{gf}^{global} \Delta_f \Leftrightarrow -\left(\mathbf{K}_{ef}^{global}\right)^{-1} \hat{\mathbf{K}}_{gf}^{global} \Delta_f = \frac{1}{\lambda} \Delta_f$$

where the subscript f indicates that the respective matrices and vectors relate to the free degrees of freedom only.

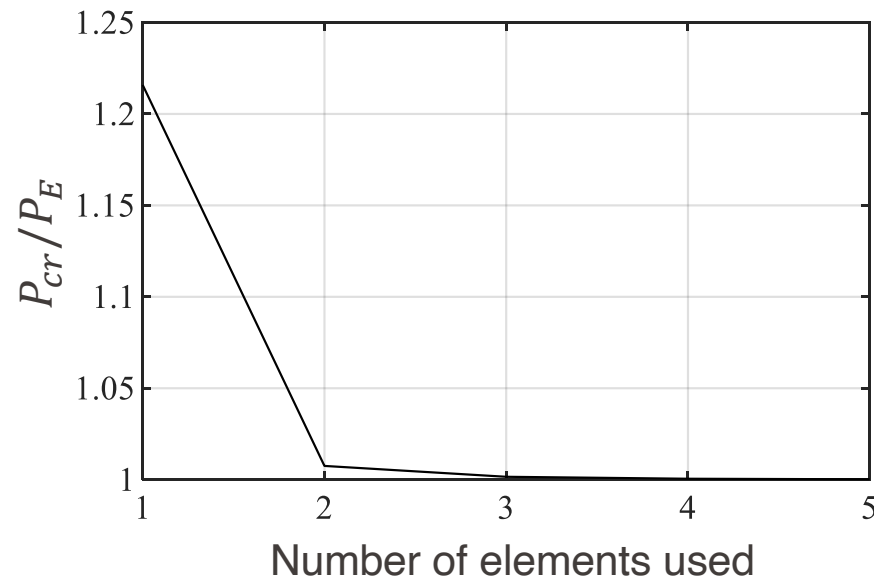
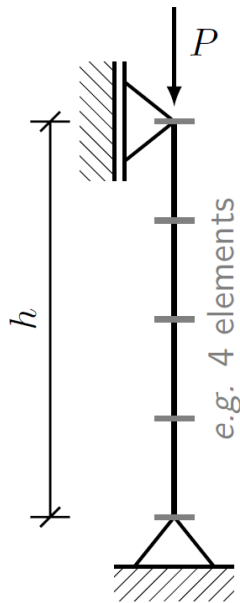
Solution of the eigenvalue problem (2)

- The following approach may be used to determine the elastic critical loads of a structure:
 1. Assemble the global elastic stiffness matrix of the structure $\mathbf{k}_e^{global}$
 2. Impose the boundary conditions (external unit loads $\mathbf{F}$)
 3. Using the static condensation, solve the system $\mathbf{F} = \mathbf{K}_e^{global} \mathbf{v}$ to determine the nodal displacements $\mathbf{v}$
 4. For each element:
 - a. In the local reference system, solve the system $\mathbf{Q} = \mathbf{k}_e^{local} \mathbf{u}$ to determine the internal forces $\mathbf{Q}$
 - b. Compute the local geometric stiffness matrix $\mathbf{k}_g^{local}$
 5. Assemble the global geometric stiffness matrix of the structure $\mathbf{k}_g^{global}$
 6. Solve the eigenvalue problem $\left(\mathbf{k}_{ef}^{global}\right)^{-1} \mathbf{k}_g^{global} \Delta_f = \frac{1}{\lambda} \Delta_f$ to determine the load multipliers λ
 7. The critical load $\mathbf{P}_{cr}$ is obtained by taking the minimum (in absolute value) load multiplier $\lambda_{min} = \min(|\lambda|)$ and multiplying it with the applied unit load $\mathbf{F}$:

$$\mathbf{P}_{cr} = \lambda_{min} \mathbf{F}$$

EPFL Example: Euler buckling load

- Determine the Euler load of a $h = 3.0m$ high Euler column. Assume a HEA 320 steel cross section ($E = 200 \text{ GPa}$, $I = 229.3 \cdot 10^6 \text{ mm}^4$ and $A = 12400 \text{ mm}^2$)
- Use 2d elastic beam elements (see previous slides for elastic and geometric stiffness matrices)
- Compare with the theoretical Euler buckling load P_E given by, $P_E = \frac{\pi^2 EI}{h^2}$



EPFL Example: Effect of the lateral restraint on the buckling load

- What is the influence of the spring stiffness on the buckling response of the column?
- The buckling load P_{cr} is given by

$$P_{cr} = \frac{\pi^2 EI}{(Kh)^2}$$

Where k is the effective length factor (Kh is the buckling length of the column)

- The stiffness k of the spring is taken as

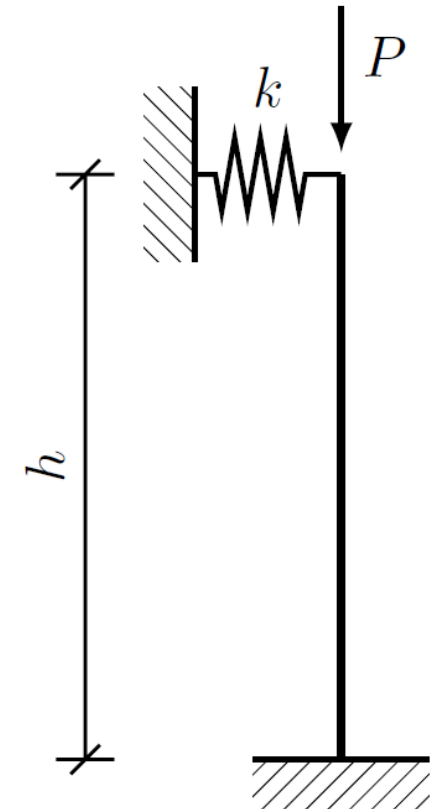
$$k = \alpha \frac{3EI}{h^3}$$

Where $3EI/h^3$ is the lateral (translational) stiffness of the cantilever column

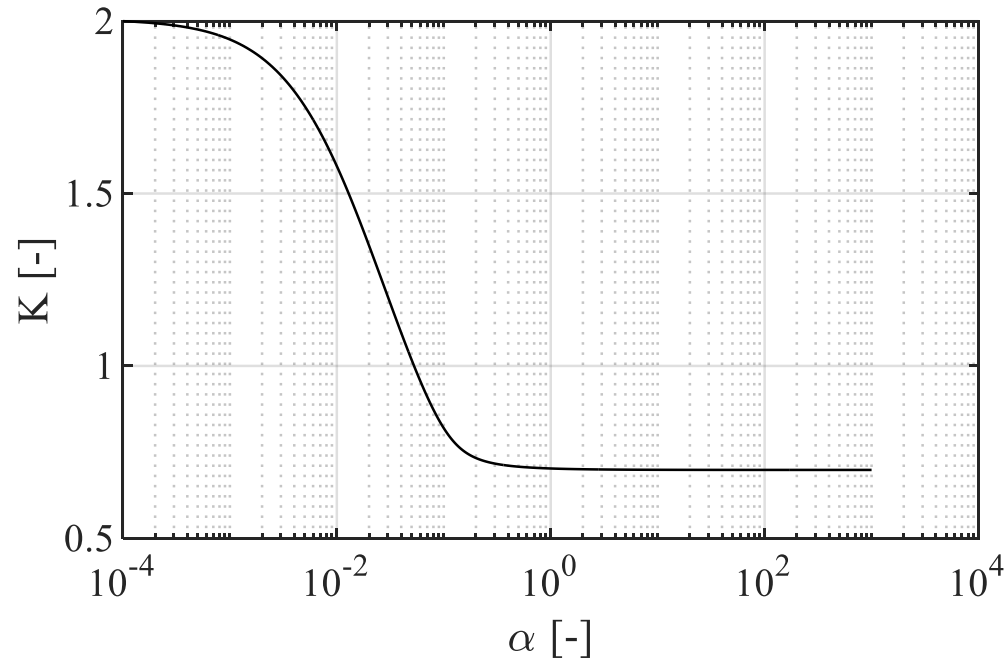
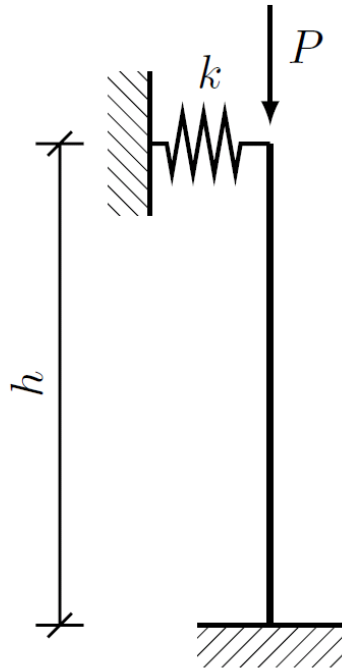
- Include the effect of the spring when assembling the global elastic stiffness matrix
- To determine the influence of the spring stiffness on the buckling length when computing the buckling load, compute the effective length factor K using

$$K = \sqrt{\frac{\pi^2 EI}{P_{cr} h^2}}$$

Where P_{cr} is obtained by solving the eigenvalue problem

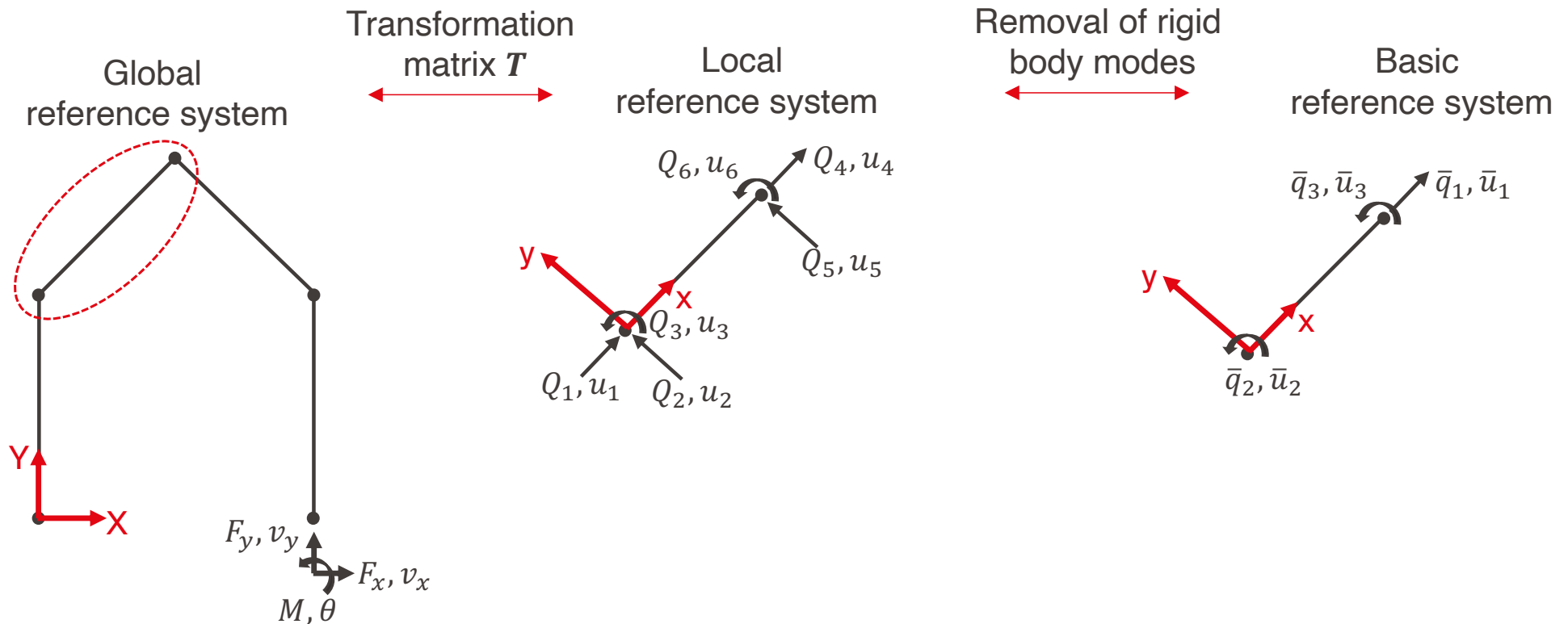


EPFL Example: Effect of the lateral restraint on the buckling load (2)



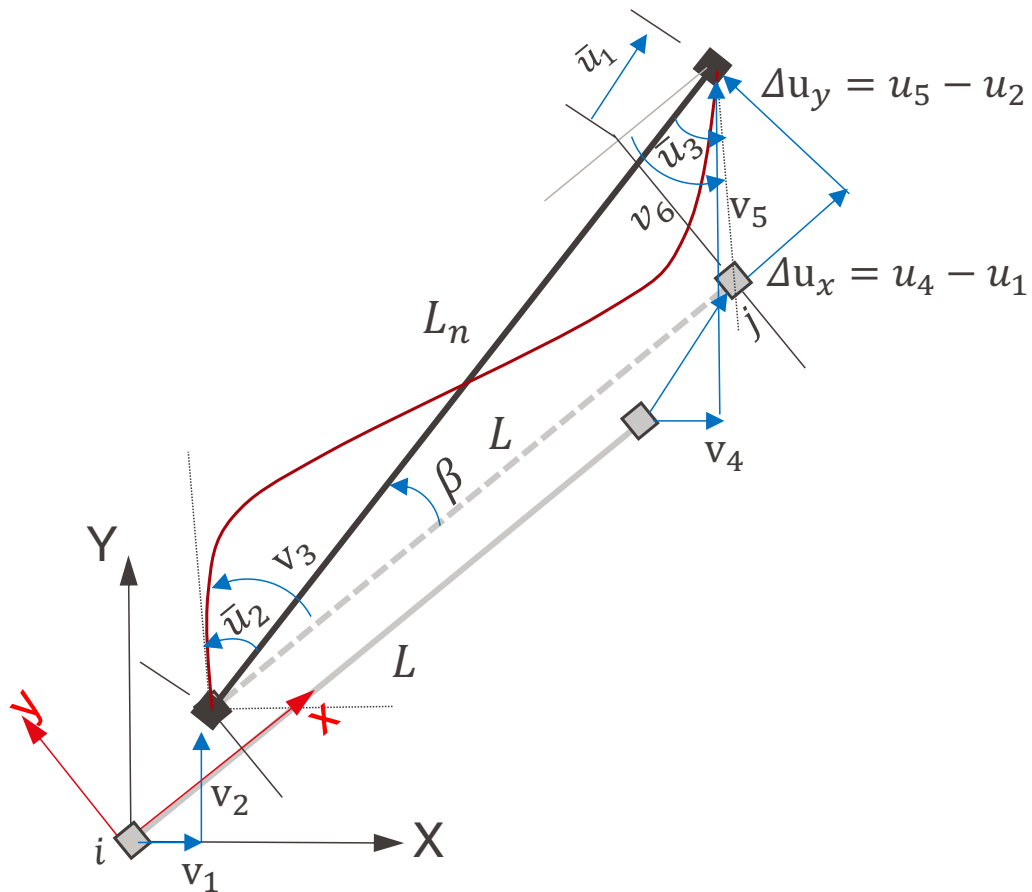
- The effective length factor K varies from $K = 2$ to $K = 0,7$
- The value $K = 2$ corresponds to a fixed-free column
- The value $K = 0,7$ corresponds to a fixed-fixed column

Basic reference system for frame elements

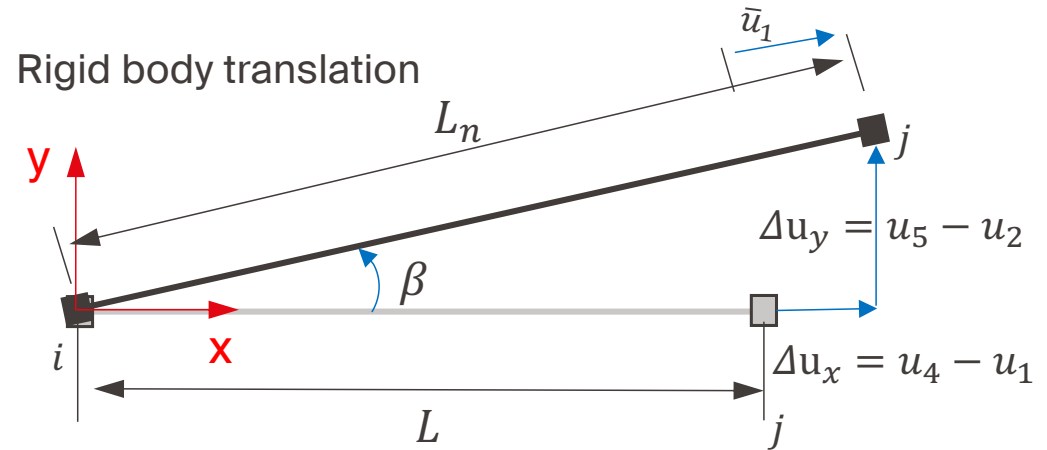


- Conventional frame (beam-column) elements are formulated within the basic reference frame

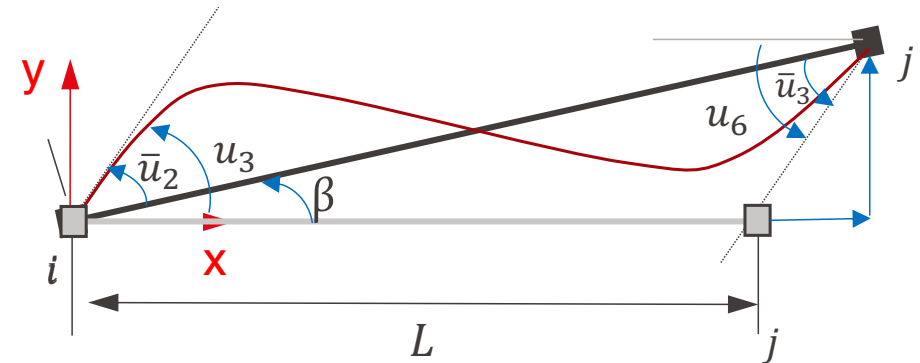
Removal of rigid body modes



Rigid body translation



Deformations in the basic reference frame



EPFL Displacement–deformation relation: Large displacements

- The displacements in the basic reference frame are given by

$$\begin{aligned}\bar{u}_1 &= L_n - L \\ \bar{u}_2 &= u_3 - \beta \\ \bar{u}_3 &= u_6 - \beta\end{aligned}$$

With

$$\beta = \arctan\left(\frac{\Delta u_y}{L + \Delta u_x}\right), L_n = \sqrt{(L + \Delta u_x)^2 + (\Delta u_y)^2}$$

EPFL Linear geometry approximation

- Assuming small deformations and rotations, expanding the arctan function using a Taylor series approximation about the point $\Delta u_x = 0, \Delta u_y = 0$,

$$\beta = \frac{\Delta u_y}{L} \left[1 - \frac{\Delta u_x}{L} + \dots \right]$$

Similarly,

$$\bar{u}_1 = L \left[\frac{\Delta u_x}{L} + \frac{1}{2} \left(\frac{\Delta u_x}{L} \right)^2 + \frac{1}{2} \left(\frac{\Delta u_y}{L} \right)^2 \right]$$

- Assuming small deformations and rotations, the second order terms can be neglected,

$$\beta = \frac{\Delta u_y}{L}$$

And

$$\bar{u}_1 = \Delta u_x$$

EPFL Linear geometry approximation (2)

- Assuming linear geometry (i.e., small deformations and rotations), the compatibility relations between the element deformations in the basic reference frame $\bar{\mathbf{u}}$ and the element displacements in the local reference frame $\mathbf{u}$ become linear:

$$\begin{aligned}\bar{u}_1 &= \Delta u_x = u_4 - u_1 = -1u_1 + 0u_2 + 0u_3 + 1u_4 + 0u_5 + 0u_6 \\ \bar{u}_2 &= u_3 - \frac{\Delta u_y}{L} = u_3 - \frac{u_5 - u_2}{L} = 0u_1 + \frac{1}{L}u_2 + 1u_3 + 0u_4 - \frac{1}{L}u_5 + 0u_6 \\ \bar{u}_3 &= u_6 - \frac{\Delta u_y}{L} = u_6 - \frac{u_5 - u_2}{L} = 0u_1 + \frac{1}{L}u_2 + 0u_3 + 0u_4 - \frac{1}{L}u_5 + 1u_6\end{aligned}$$

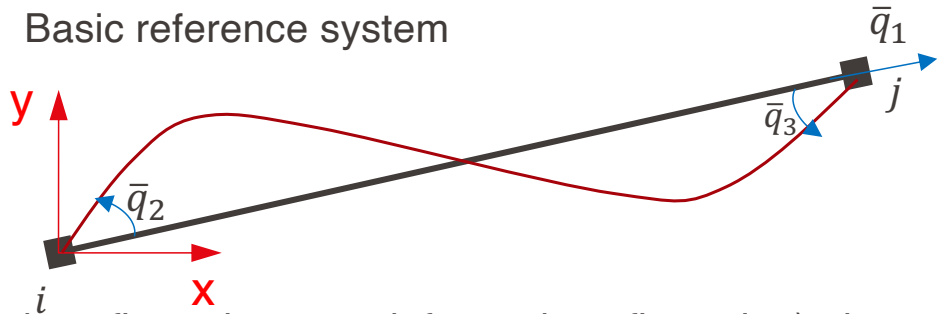
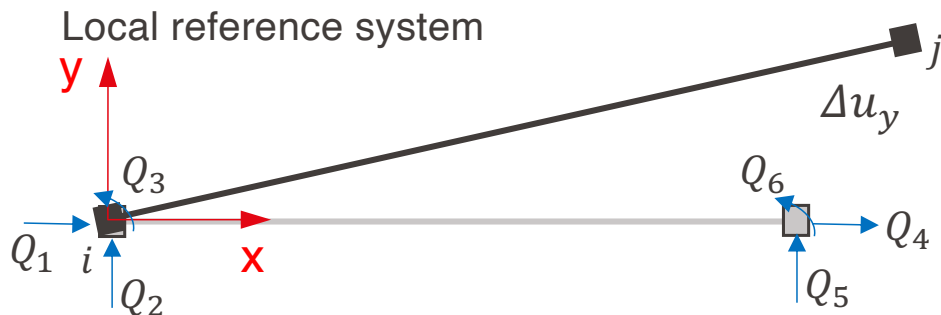
- In matrix form

$$\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{pmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{L} & 1 & 0 & -\frac{1}{L} & 0 \\ 0 & \frac{1}{L} & 0 & 0 & -\frac{1}{L} & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}$$

In compact form

$$\bar{\mathbf{u}} = \mathbf{L}\mathbf{u}$$

EPFL Linear geometry approximation (3)



- Using the equilibrium of forces (linear geometry $\rightarrow$ deformed configuration $\approx$ undeformed configuration), the element resisting forces in the local reference system are given by

$$\begin{aligned} Q_4 &= \bar{q}_1 \\ Q_3 &= \bar{q}_2 \\ Q_6 &= \bar{q}_3 \\ Q_1 &= -Q_4 = -\bar{q}_1 \\ Q_2 &= \frac{1}{L}(Q_3 + Q_6) = \frac{1}{L}(\bar{q}_2 + \bar{q}_3) \\ Q_5 &= \frac{-1}{L}(Q_3 + Q_6) = \frac{-1}{L}(\bar{q}_2 + \bar{q}_3) \end{aligned}$$

- In matrix form

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{pmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/L & 1/L \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1/L & -1/L \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \end{pmatrix} \text{ or } \mathbf{Q} = \mathbf{L}^T \bar{\mathbf{q}}$$

EPFL Linear geometry approximation (4)

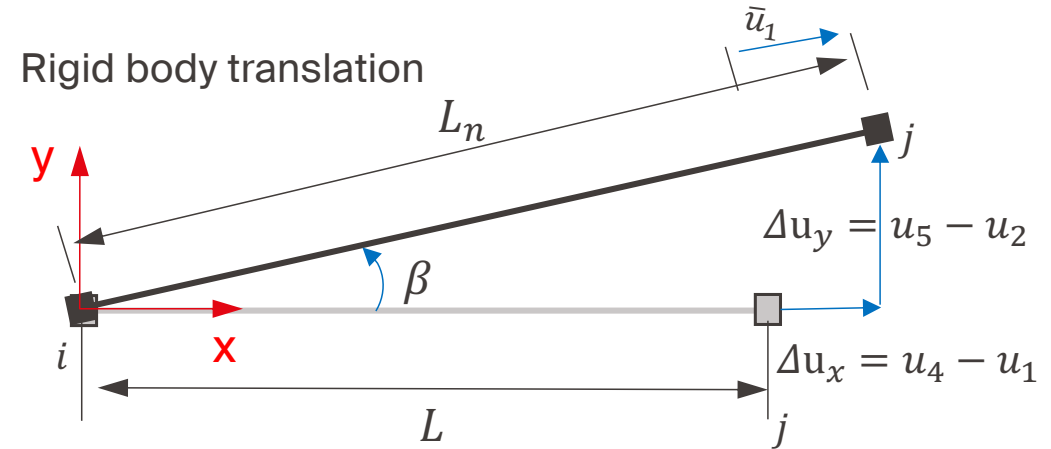
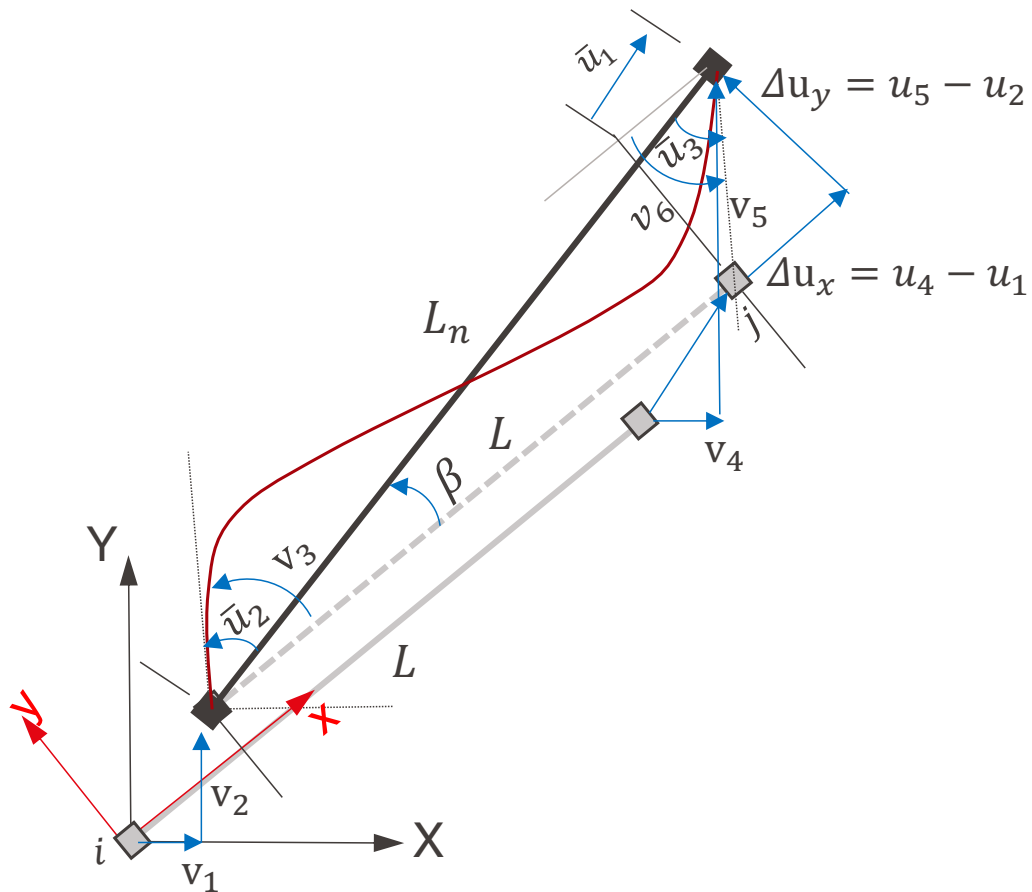
- The element stiffness matrix in the basic reference system, $\bar{\mathbf{k}}$ is given by

$$\bar{\mathbf{k}} = \frac{\partial \bar{\mathbf{q}}}{\partial \bar{\mathbf{u}}}$$

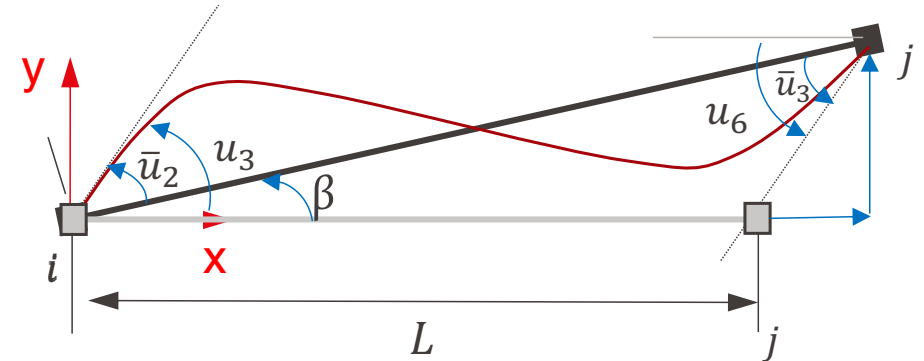
- The element stiffness matrix in the local reference system, $\mathbf{k}_{local}$ is given by

$$\mathbf{k}_{local} = \frac{\partial \mathbf{Q}}{\partial \mathbf{u}} = \frac{\partial}{\partial \mathbf{u}} (\mathbf{L}^T \bar{\mathbf{q}}) = \frac{\partial}{\partial \mathbf{u}} (\mathbf{L}^T \bar{\mathbf{k}} \bar{\mathbf{u}}) = \frac{\partial}{\partial \mathbf{u}} (\mathbf{L}^T \bar{\mathbf{k}} \mathbf{L} \mathbf{u}) = \mathbf{L}^T \bar{\mathbf{k}} \mathbf{L}$$

Nonlinear geometry: Large displacements and rotations



Deformations in the basic reference frame



EPFL Displacement–deformation relation: Large displacements

- The displacements in the basic reference frame are given by

$$\bar{u}_1 = L_n - L$$

$$\bar{u}_2 = u_3 - \beta$$

$$\bar{u}_3 = u_6 - \beta$$

With

$$\beta = \arctan\left(\frac{\Delta u_y}{L + \Delta u_x}\right), L_n = \sqrt{(L + \Delta u_x)^2 + (\Delta u_y)^2}$$

- Large displacements and rotations → Corotational formulation
- Define

$$c = \cos(\beta) = \frac{L + \Delta u_x}{L_n}, s = \sin(\beta) = \frac{\Delta u_y}{L_n}$$

Corotational formulation: Virtual displacements

- The virtual basic displacements are obtained through differentiation of the previous equations

$$\begin{aligned}\delta \bar{u}_1 &= \delta L_n = \frac{1}{2L_n} (2\delta \Delta u_x (L + \Delta u_x) + 2\Delta u_y \delta \Delta u_y) = c(\delta u_4 - \delta u_1) + s(\delta u_5 - \delta u_2) \\ &= [-c \quad -s \quad 0 \quad c \quad s \quad 0] \delta \mathbf{u}\end{aligned}$$

$$\delta \bar{u}_2 = \delta u_3 - \delta \beta$$

$$\delta \bar{u}_3 = \delta u_6 - \delta \beta$$

Where,

$$\delta \beta = \frac{1}{1 + \left(\frac{\Delta u_y}{L + \Delta u_x} \right)^2} \left(\frac{\delta \Delta u_y (L + \Delta u_x) - \Delta u_y \delta \Delta u_y}{(L + \Delta u_x)^2} \right)$$

And after simplification

$$\delta \beta = \frac{1}{L_n} [c(\delta u_5 - \delta u_2) - s(\delta u_4 - \delta v_1)] = \frac{1}{L_n} [s \quad -c \quad 0 \quad -s \quad c \quad 0] \delta \mathbf{u}$$

Corotational formulation: Virtual displacements (2)

- The relations between the displacements in the local reference frame $\mathbf{u}$ and the displacements in the basic reference frame $\bar{\mathbf{u}}$ are given by,

$$\begin{cases} \delta \bar{u}_1 = [-c & -s & 0 & c & s & 0] \delta \mathbf{u} \\ \delta \bar{u}_2 = \delta u_3 - \frac{1}{L_n} [s & -c & 0 & -s & c & 0] \delta \mathbf{u} \\ \delta \bar{u}_3 = \delta u_6 - \frac{1}{L_n} [s & -c & 0 & -s & c & 0] \delta \mathbf{u} \end{cases}$$

In matrix form:

$$\delta \bar{\mathbf{u}} = \mathbf{L} \delta \mathbf{u}$$

With

$$\mathbf{L} = \begin{bmatrix} -c & -s & 0 & c & s & 0 \\ -s/L_n & c/L_n & 1 & s/L_n & -c/L_n & 0 \\ -s/L_n & c/L_n & 0 & s/L_n & -c/L_n & 1 \end{bmatrix}$$

Corotational formulation: Resisting forces

- The relation between the resisting forces in the basic reference system $\bar{\mathbf{q}}$ and the resisting forces in the local reference system $\mathbf{Q}$ is obtained by equating the virtual work in both basic and local reference systems,

$$W = \delta \mathbf{u}^T \mathbf{Q} = \delta \bar{\mathbf{u}}^T \bar{\mathbf{q}} = \delta \mathbf{v}^T \mathbf{L}^T \bar{\mathbf{q}}$$

- This equation must hold true for any arbitrary $\delta \mathbf{u}^T$, therefore, the element resisting forces in the local reference frame $\mathbf{Q}$ are given by

$$\mathbf{Q} = \mathbf{L}^T \bar{\mathbf{q}}$$

- The element resisting forces in the basic reference frame $\bar{\mathbf{q}}$ depend on the element formulation and will be discussed in the following weeks.

Corotational formulation: Tangent stiffness matrix

- The tangent stiffness in the local reference frame $\mathbf{K}_l$ defined by

$$\delta \mathbf{Q} = \mathbf{K}_l \delta \mathbf{u}$$

is obtained by differentiating the relation $\mathbf{Q} = \mathbf{L}^T \bar{\mathbf{q}}$, which gives

$$\delta \mathbf{Q} = \mathbf{L}^T \delta \bar{\mathbf{q}} + \delta \mathbf{L}^T \bar{\mathbf{q}} = \mathbf{L}^T \delta \bar{\mathbf{q}} + \bar{q}_1 \delta \mathbf{L}_1 + \bar{q}_2 \delta \mathbf{L}_2 + \bar{q}_3 \delta \mathbf{L}_3$$

Where $\mathbf{L}_k$ is the k^{th} column of $\mathbf{L}^T$

- Introduce the quantities

$$\begin{aligned} \mathbf{r} &= [-c \quad -s \quad 0 \quad c \quad s \quad 0]^T \\ \mathbf{z} &= [s \quad -c \quad 0 \quad -s \quad c \quad 0]^T \end{aligned}$$

And differentiate them

$$\delta \mathbf{r} = \mathbf{z} \delta \beta, \delta \mathbf{z} = -\mathbf{r} \delta \beta$$

These can be used to rewrite

$$\delta \bar{u}_1 = \delta L_n = \mathbf{r}^T \delta \mathbf{u}, \quad \delta \beta = \frac{\mathbf{z}^T}{L_n} \delta \mathbf{u}$$

EPFL Corotational formulation: Tangent stiffness matrix (2)

- Using these notations

$$\begin{aligned}\mathbf{L}_1 &= \mathbf{r} \\ \mathbf{L}_2 &= [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^T - \frac{\mathbf{z}}{L_n} \\ \mathbf{L}_3 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T - \frac{\mathbf{z}}{L_n}\end{aligned}$$

Which by differentiation give

$$\begin{aligned}\delta \mathbf{L}_1 &= \delta \mathbf{r} = \frac{\mathbf{z}\mathbf{z}^T}{L_n} \delta \mathbf{u} \\ \delta \mathbf{L}_2 &= \delta \mathbf{L}_3 = -\frac{\delta \mathbf{z}}{L_n} + \frac{\mathbf{z}\delta L_n}{L_n^2} = \frac{1}{L_n^2} (\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T) \delta \mathbf{u}\end{aligned}$$

EPFL Corotational formulation: Tangent stiffness matrix (3)

- The tangent stiffness matrix in the basic reference system $\bar{\mathbf{k}}$ defined by

$$\delta \bar{\mathbf{q}} = \bar{\mathbf{k}} \delta \bar{\mathbf{u}} = \bar{\mathbf{k}} \mathbf{L} \delta \mathbf{u}$$

- Finally

$$\begin{aligned} \delta \mathbf{Q} &= \mathbf{K}_l \delta \mathbf{u} = \mathbf{L}^T \delta \bar{\mathbf{q}} + \bar{q}_1 \delta \mathbf{L}_1 + \bar{q}_2 \delta \mathbf{L}_2 + \bar{q}_3 \delta \mathbf{L}_3 \\ &= \mathbf{L}^T \bar{\mathbf{k}} \mathbf{L} \delta \mathbf{u} + \bar{q}_1 \frac{\mathbf{z} \mathbf{z}^T}{L_n} \delta \mathbf{u} + \bar{q}_2 \frac{1}{L_n^2} (\mathbf{r} \mathbf{z}^T + \mathbf{z} \mathbf{r}^T) \delta \mathbf{u} + \bar{q}_3 \frac{1}{L_n^2} (\mathbf{r} \mathbf{z}^T + \mathbf{z} \mathbf{r}^T) \delta \mathbf{u} \\ &= \left(\mathbf{L}^T \bar{\mathbf{k}} \mathbf{L} + \bar{q}_1 \frac{\mathbf{z} \mathbf{z}^T}{L_n} + \bar{q}_2 \frac{1}{L_n^2} (\mathbf{r} \mathbf{z}^T + \mathbf{z} \mathbf{r}^T) + \bar{q}_3 \frac{1}{L_n^2} (\mathbf{r} \mathbf{z}^T + \mathbf{z} \mathbf{r}^T) \right) \delta \mathbf{u} \end{aligned}$$

- The tangent stiffness in the local reference frame $\mathbf{K}_l$ is given by

$$\mathbf{K}_l = \underbrace{\mathbf{L}^T \bar{\mathbf{k}} \mathbf{L}}_{\text{Material stiffness matrix}} + \underbrace{\bar{q}_1 \frac{\mathbf{z} \mathbf{z}^T}{L_n} + \frac{1}{L_n^2} (\mathbf{r} \mathbf{z}^T + \mathbf{z} \mathbf{r}^T) (\bar{q}_2 + \bar{q}_3)}_{\text{Geometric stiffness matrix}}$$

- The element tangent stiffness matrix in the basic reference frame $\bar{\mathbf{k}}$ depends on the element formulation. This will be discussed in the coming weeks.

EPFL Corotational formulation: Geometric stiffness matrix

- The geometric stiffness matrix $\mathbf{k}_{geom}$ is computed using

$$\mathbf{k}_{geom} = \bar{q}_1 \frac{\mathbf{z}\mathbf{z}^T}{L_n} + \frac{1}{L_n^2} (\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T) (\bar{q}_2 + \bar{q}_3)$$

$$= \frac{\bar{q}_1}{L_n} \begin{bmatrix} s^2 & -cs & 0 & -s^2 & cs & 0 \\ -cs & c^2 & 0 & cs & -c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & cs & 0 & s^2 & -cs & 0 \\ cs & -c^2 & 0 & -cs & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\bar{q}_2 + \bar{q}_3}{L_n^2} \begin{bmatrix} -2sc & c^2 - s^2 & 0 & 2sc & -c^2 + s^2 & 0 \\ c^2 - s^2 & 2cs & 0 & -c^2 + s^2 & -2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2sc & -c^2 + s^2 & 0 & -2sc & c^2 - s^2 & 0 \\ -c^2 + s^2 & -2cs & 0 & c^2 - s^2 & 2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Transformation to the global reference system

- From the previous lectures, to transform the stiffness matrix from the local reference frame $\mathbf{K}_l$ to the stiffness matrix in the global reference system $\mathbf{K}_g$, the transformation matrix $\mathbf{T}$ is used:

$$\mathbf{K}_g = \mathbf{T}^T \mathbf{K}_l \mathbf{T}$$

- Using the definition of the local stiffness matrix $\mathbf{K}_l$ using the stiffness matrix from the basic reference frame $\bar{\mathbf{k}}$ gives

$$\mathbf{K}_g = \mathbf{T}^T \left(\underbrace{\mathbf{L}^T \bar{\mathbf{k}} \mathbf{L}}_{\text{Material stiffness matrix}} + \underbrace{\bar{q}_1 \frac{\mathbf{z}\mathbf{z}^T}{L_n} + \frac{1}{L_n^2} (\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T)(\bar{q}_2 + \bar{q}_3)}_{\text{Geometric stiffness matrix}} \right) \mathbf{T}$$

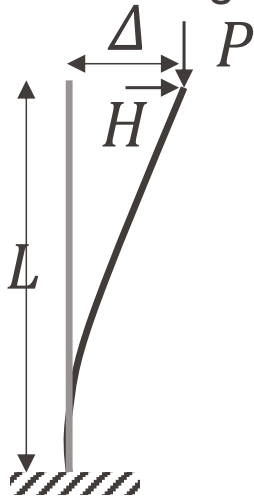
Material stiffness matrix
Geometric stiffness matrix

Tangent stiffness matrix in the local reference system

Tangent stiffness matrix in the global reference system

EPFL P-Delta effects on columns

- Consider the following column loaded laterally and vertically



1st order moment
(Linear geometry)

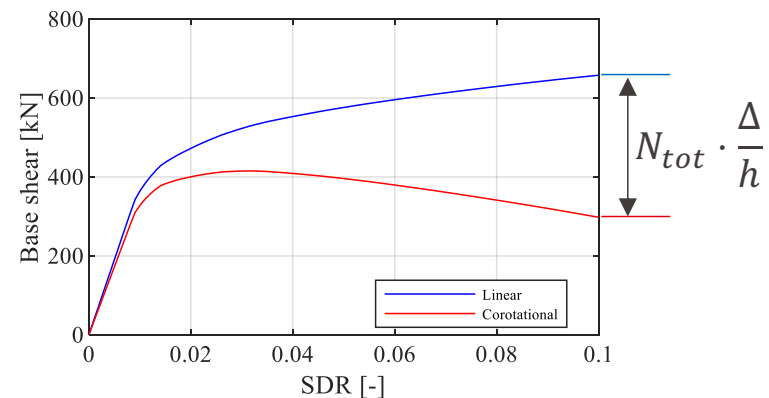
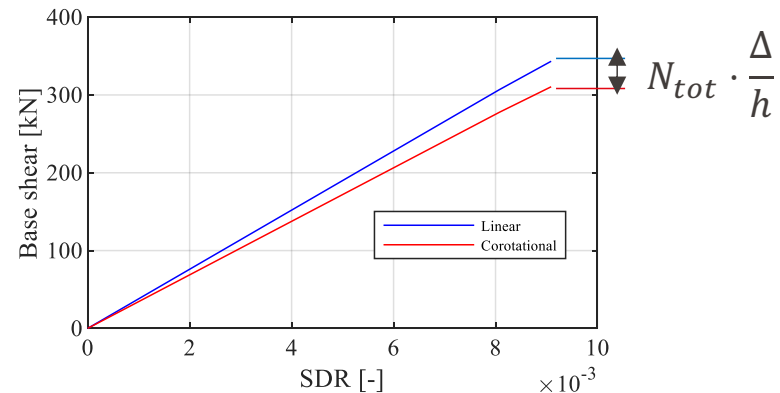
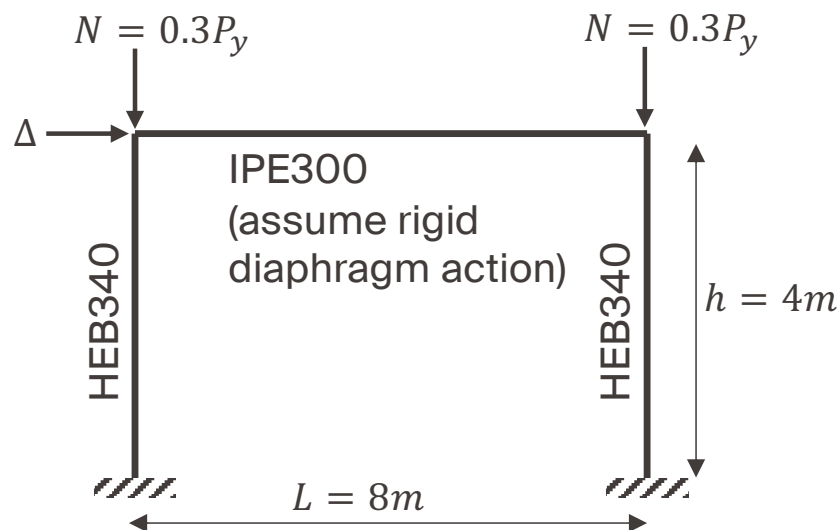


2nd order moment
(Nonlinear geometry)

- Assume linear geometry: P-Delta effects are not captured
- Assuming a linear versus nonlinear geometry strongly influences the simulation results

EPFL Nonlinear geometric effects on the response of frames

- Assuming a linear or nonlinear geometry is very important when modeling frame structures.



- For the simulations, a 2d force-based beam-column element was used and the UVC (i.e., isotropic hardening rule) constitutive material law. These will be explained later.