CIVIL 449: Nonlinear Analysis of Structures

School of Architecture, Civil & Environmental Engineering
Civil Engineering Institute

Revision on Matrix Structural Analysis

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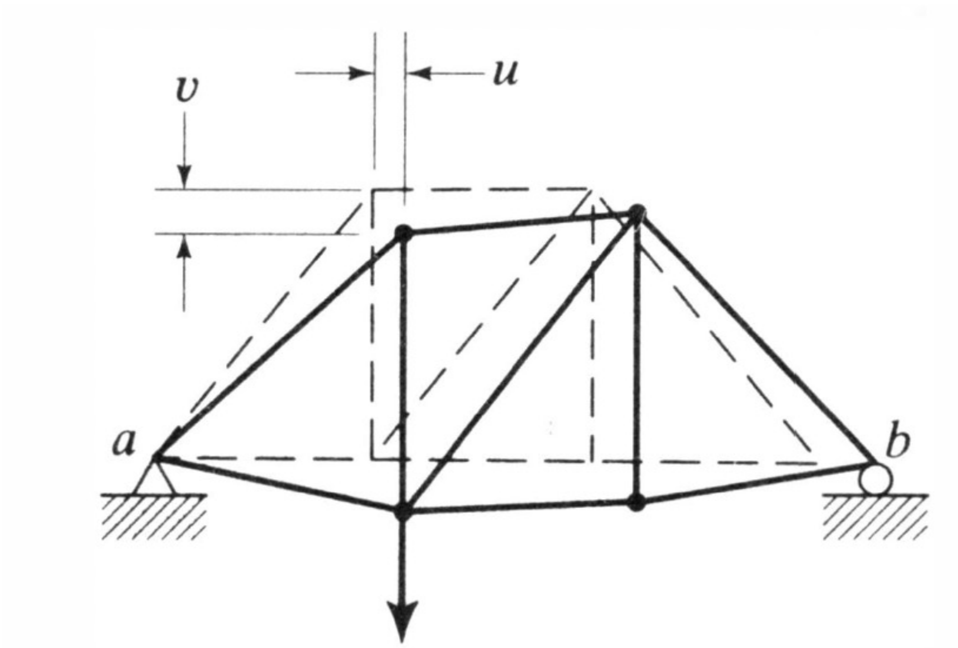
Objectives of today's lecture

To introduce:

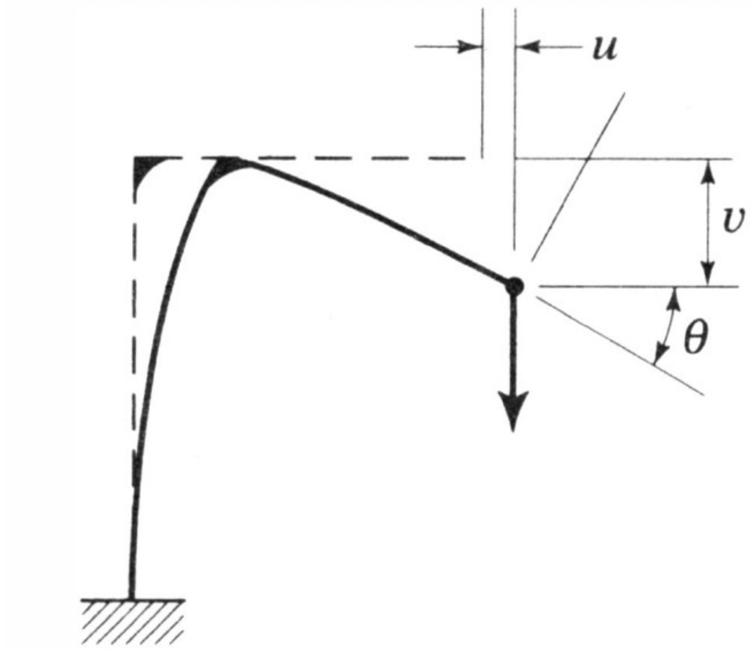
- Definitions and concepts
- Coordinate system
- Direct stiffness method
- Static condensation
- Examples of application

Degrees of freedom
-Displacement field

Pin-jointed plane truss

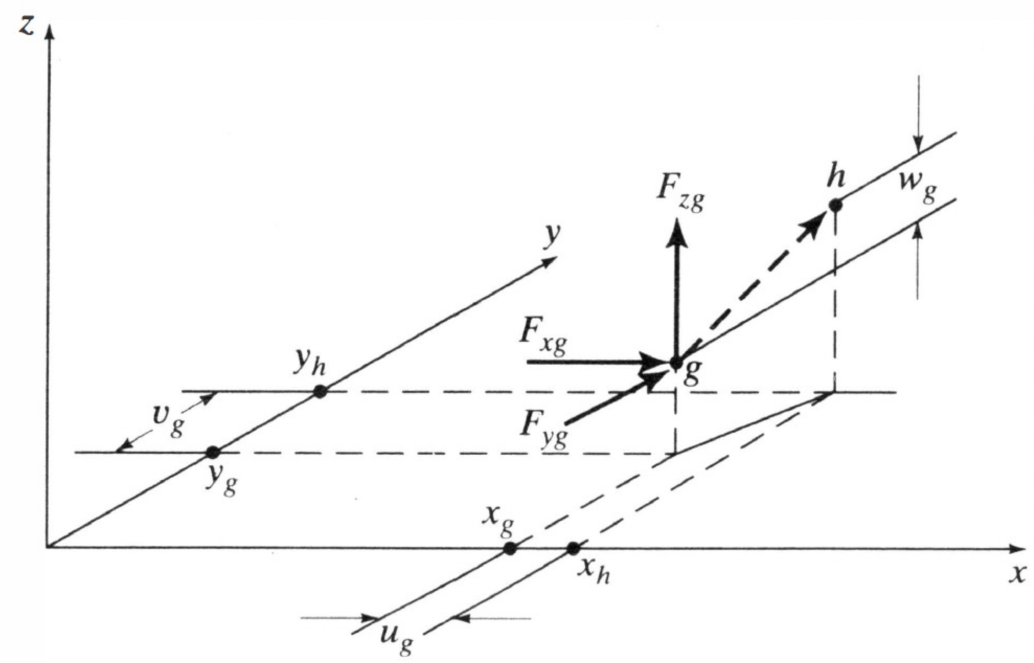


Plane frame

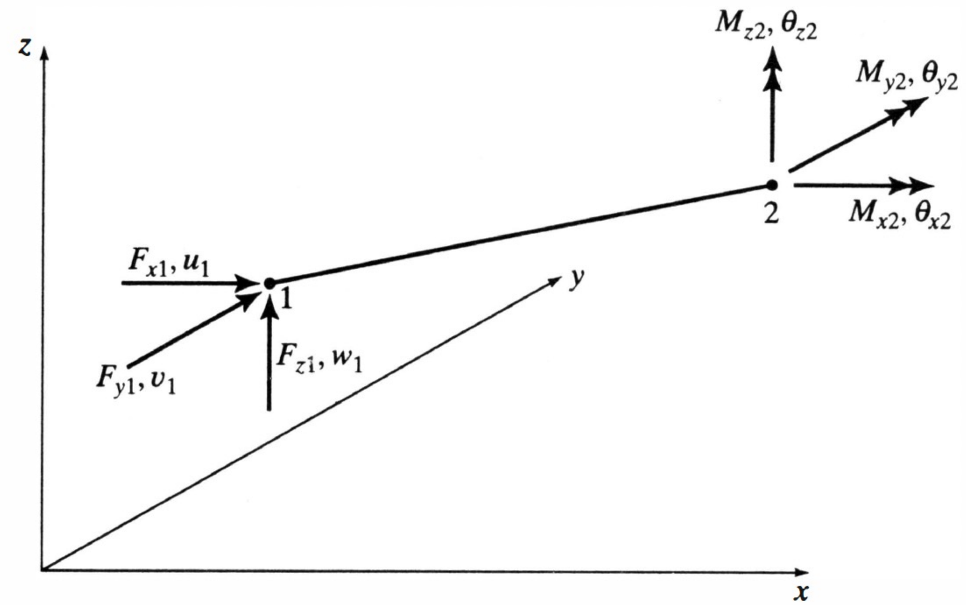


Coordinate systems and conditions of analysis

Displacement from point *g* to point *h*

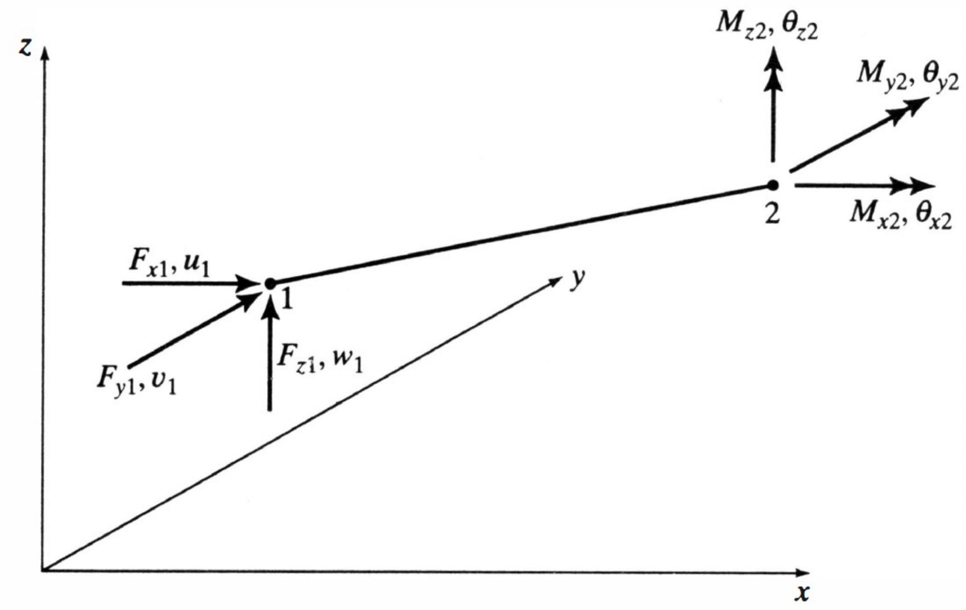


Forces, moments and corresponding displacements



$$\theta_{x2} = \left.\frac{\vartheta w}{\vartheta y}\right|_2 \qquad \theta_{y2} = \left.\frac{\vartheta w}{\vartheta x}\right|_2 \qquad \theta_{z2} = \left.\frac{\vartheta v}{\vartheta x}\right|_2$$

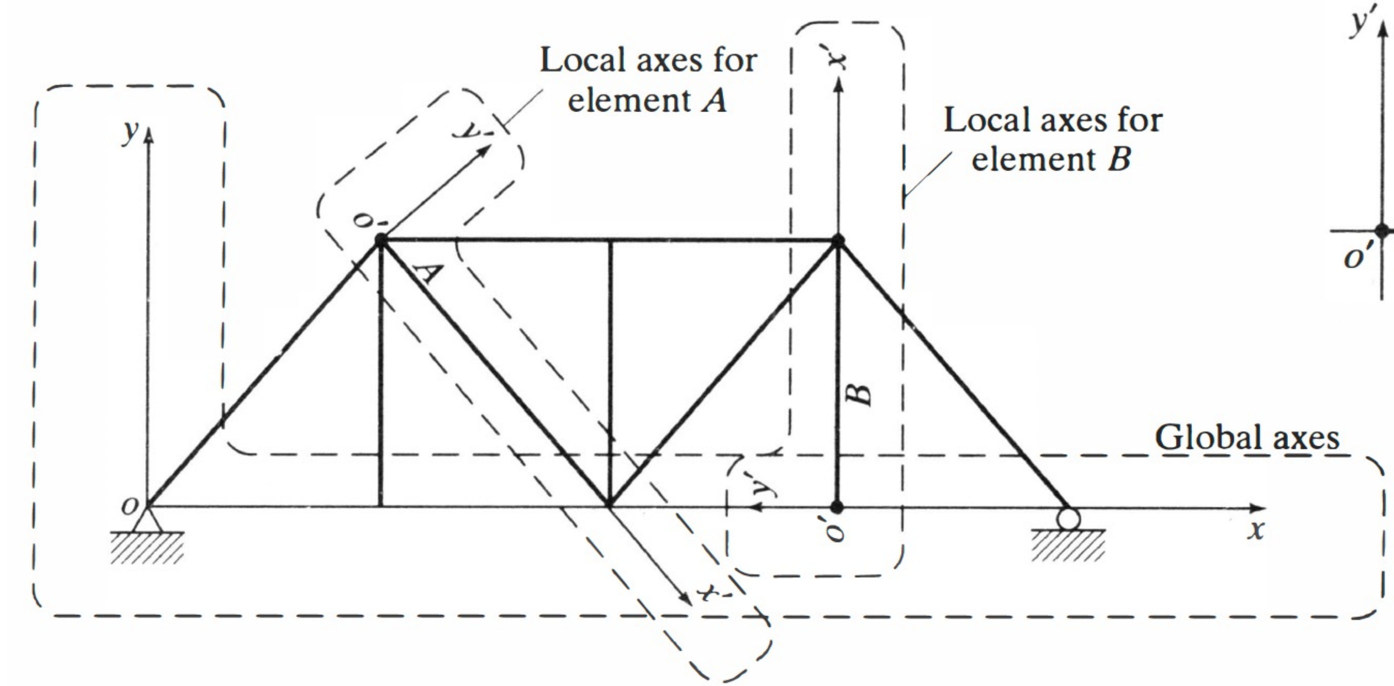
Coordinate systems and conditions of analysis (2)

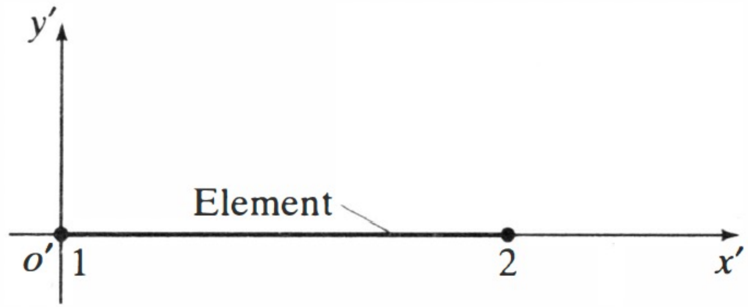


$$\{\mathbf{F}\} = \lfloor F_{x1} \quad F_{y1} \quad F_{z1} \quad M_{x2} \quad M_{y2} \quad M_{z2} \rfloor^{\mathrm{T}}$$

$$\{\mathbf{\Delta}\} = \lfloor u_1 \quad v_1 \quad w_1 \quad \theta_{x2} \quad \theta_{y2} \quad \theta_{z2} \rfloor^{\mathrm{T}}$$

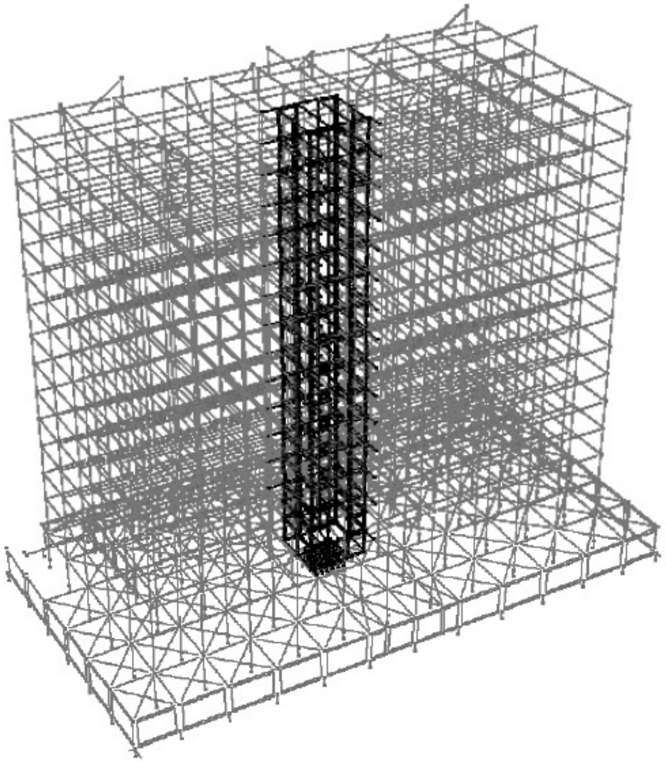
Coordinate systems and conditions of analysis (3)





Idealization of structures into frame and/or truss elements

680 Folsom Street, San Francisco, USA

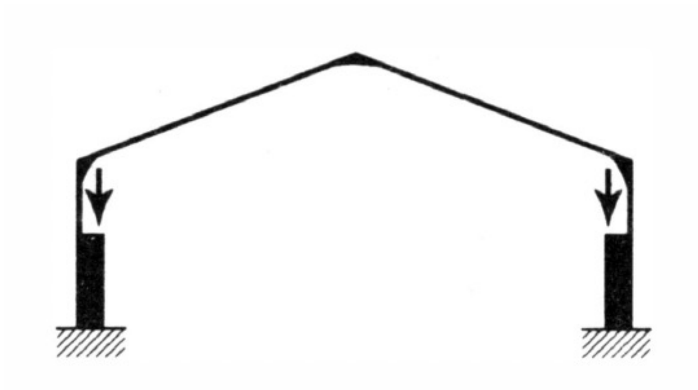


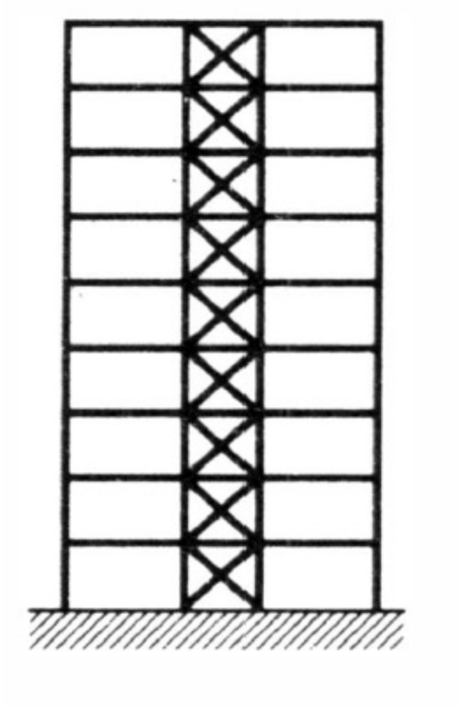
Idealization of structures into frame and/or truss elements

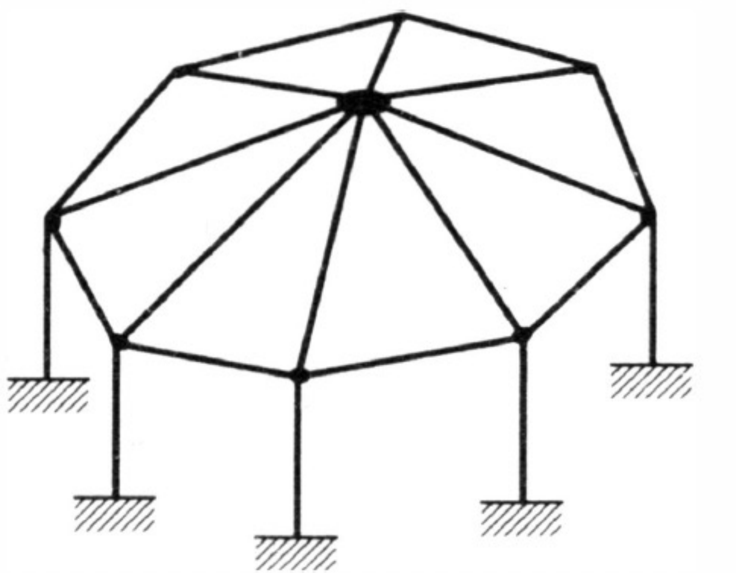
Multi-story frame

Space frame

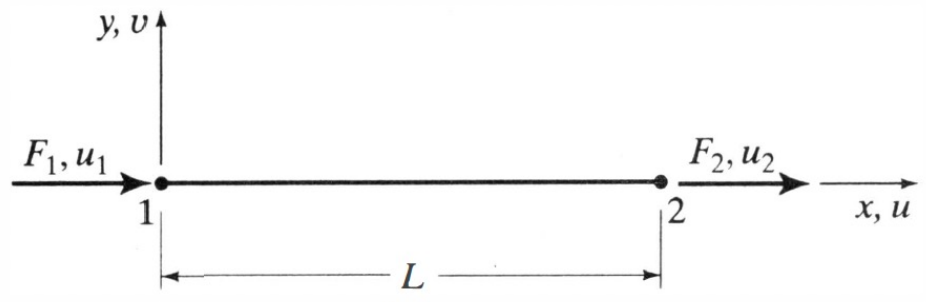
Industrial hall







Axial force element: Force-displacement relations
-Element stiffness matrix



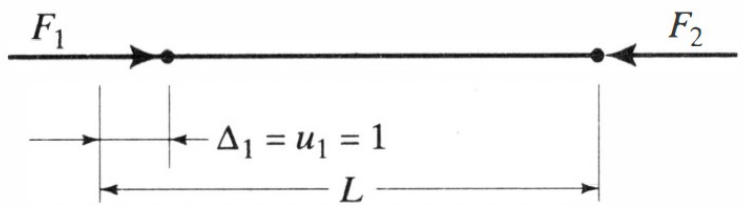
$$\{\mathbf{F}\} = [\mathbf{k}]\{\mathbf{\Delta}\}$$

$[\mathbf{k}]$ Element stiffness matrix

k_{ij} Element stiffness coefficient

Axial force element: Force-displacement relations
-Element stiffness matrix

If a displacement Δ_j of unit value is imposed and all other degrees of freedom are held fixed, the force F_i is equal to k_{ij}



$$\{\mathbf{F}\} = \lfloor F_1 \quad F_2 \rfloor^{\mathrm{T}}$$

$$\{\mathbf{k}_{i1}\} = \lfloor k_{11} \quad k_{21} \rfloor^{\mathrm{T}}$$

From basic mechanics:

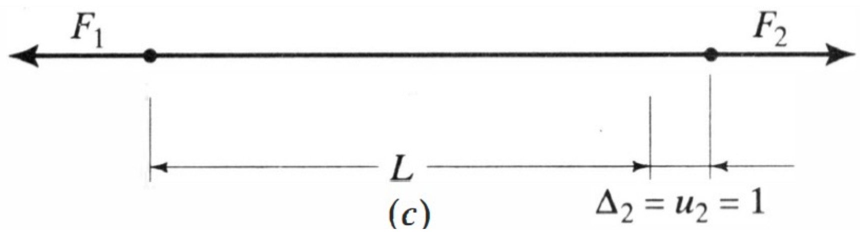
$$u_1 = \frac{F_1 L}{EA} \qquad F_1 = \frac{EA}{L} u_1$$

By equilibrium:

$$F_2 = -F_1 = -\frac{EA}{L} u_1$$

Axial force element: Force-displacement relations
-Element stiffness matrix

Similarly,



$$F_2 = -F_1 = \frac{EA}{L} u_2$$

Collected in matrix form:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \underbrace{\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{Stiffness matrix of a truss element}} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Axial force element: Force-displacement relations
-Element flexibility equations

Element flexibility equations express, for elements supported in a stable manner, the joint displacements, $\{\Delta_f\}$, as a function of the joint forces, $\{\mathbf{F}_f\}$:

$$\{\Delta_f\} = [\mathbf{d}]\{\mathbf{F}_f\}$$

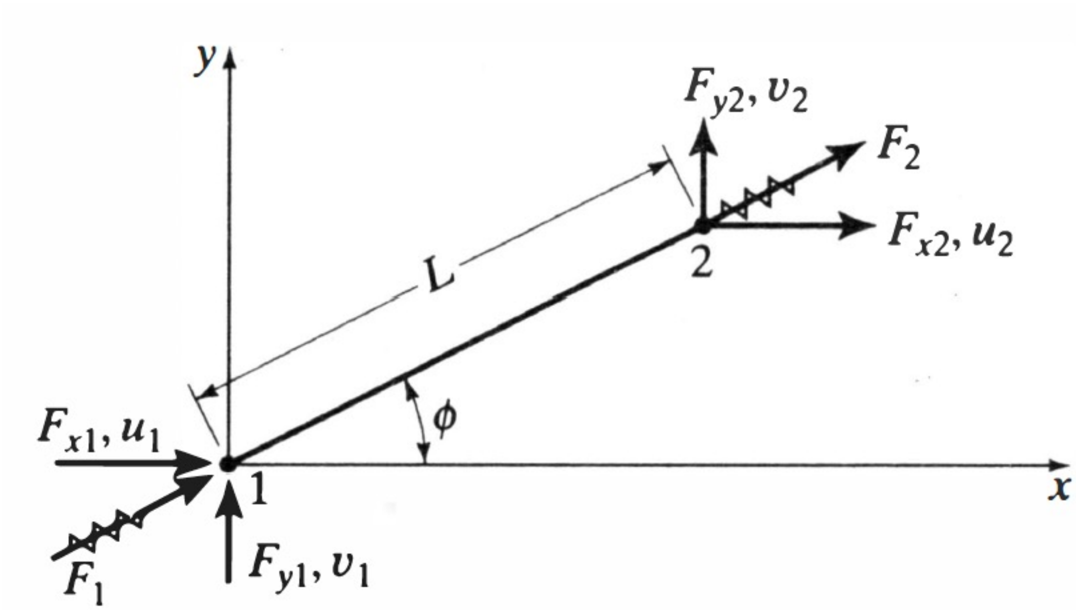
$[\mathbf{d}]$ Element flexibility matrix (inverse of $[\mathbf{k}]$)

d_{ij} Element flexibility coefficient

Axial force element: Global stiffness equations

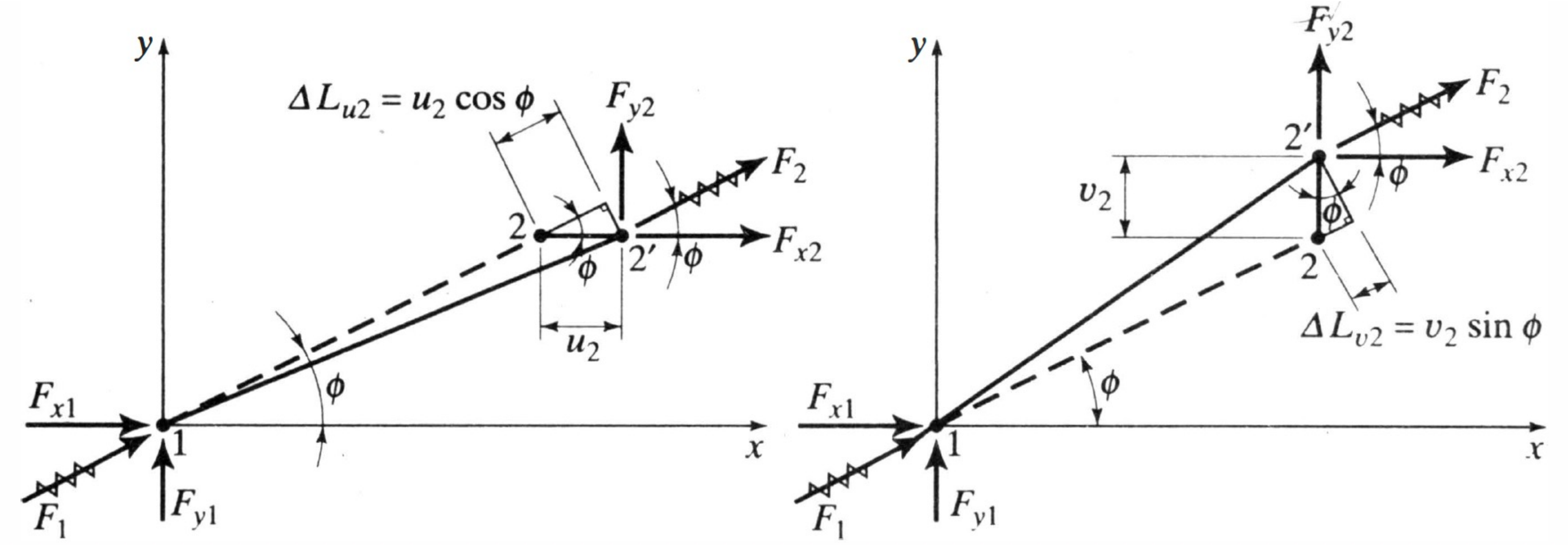
Simplest way to form the global stiffness equations for the analysis of a structure:

- start with element stiffness equations in a local coordinate system
- Transform to global axes



Axial force element: Global stiffness equations (2)

Impose a small displacement in the x and y directions of node 2



$$\Delta L_{u2} = u_2 cos\phi \qquad \Delta L_{v2} = v_2 sin\phi$$

$$F_2 = \frac{EA}{L}\Delta L_{u2} = \frac{EAcos\phi}{L}\cdot u_2 \qquad F_2 = \frac{EA}{L}\Delta L_{v2} = \frac{EAsin\phi}{L}\cdot v_2$$

Axial force element: Global stiffness equations (3)

Analyzing the force into the global coordinate system (x-direction):

$$F_{x2} = -F_{x1} = F_2 cos\phi = \frac{EA}{L}\cos^2\phi \cdot u_2$$

$$F_{y2} = -F_{y1} = F_2 sin\phi = \frac{EA}{L} sin\,\phi\, cos\phi \cdot u_2$$

Analyzing the force into the global coordinate system (y-direction):

$$F_{x2} = -F_{x1} = F_2 cos\phi = \frac{EA}{L} sin\,\phi\, cos\phi \cdot v_2$$

$$F_{y2} = -F_{y1} = F_2 sin\phi = \frac{EA}{L} sin^2\phi \cdot v_2$$

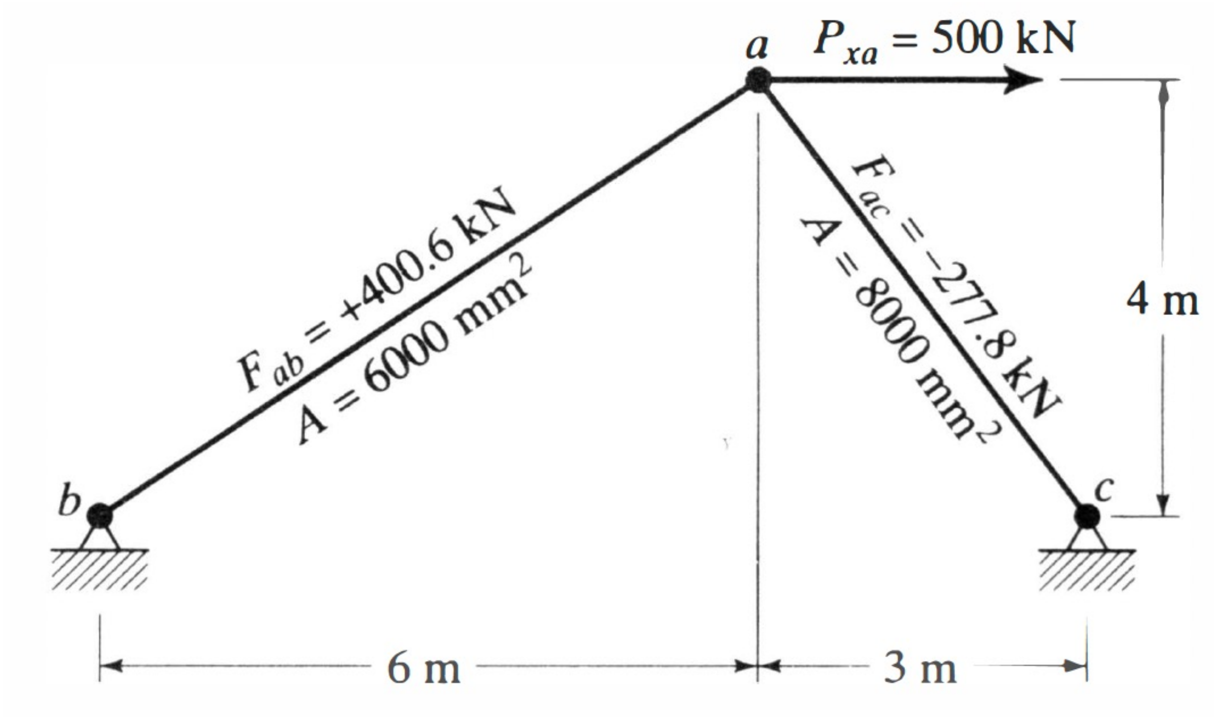
Axial force element: Global stiffness equations (4)

In a matrix form

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} \cos^2\phi & sin\phi cos\phi & -\cos^2\phi & -sin\phi cos\phi \\ sin\phi cos\phi & \sin^2\phi & -sin\phi cos\phi & -\sin^2\phi \\ -\cos^2\phi & -sin\phi cos\phi & \cos^2\phi & sin\phi cos\phi \\ -sin\phi cos\phi & -\sin^2\phi & sin\phi cos\phi & \sin^2\phi \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Example

A statically determinate truss is subjected to the load and resulting bar forces shown. What is the displacement of a? assume E=200,000MPa

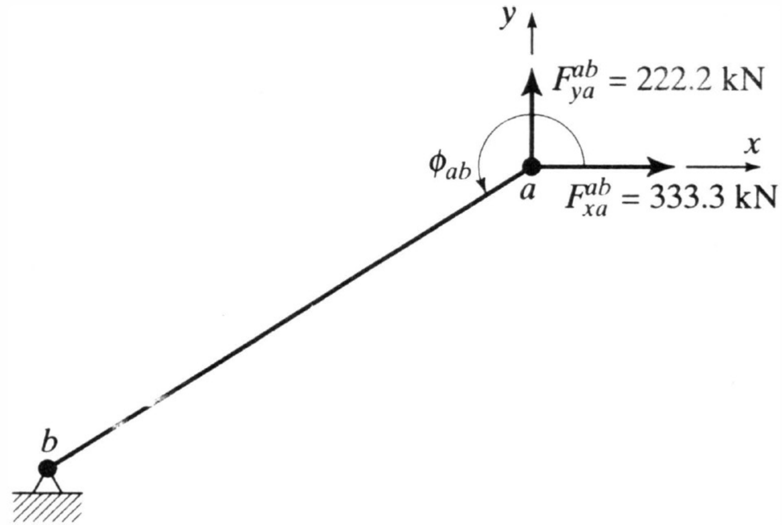


Example (2)

Consider ab:

$$\left(\frac{EA}{L}\right)_{ab} = \frac{200 \times 6 \times 10^3}{\sqrt{6^2 + 4^2} \times 10^3} = 166.4 \text{ kN/mm}$$

$$\phi_{ab} = \tan^{-1}\left(\frac{-4}{-6}\right) = 213.69°$$



$$F_{xa}^{ab} = 166.4(\cos^2\phi_{ab} \cdot u_a + \sin\phi_{ab}\cos\phi_{ab} \cdot v_a)$$

$$333.3 = 166.4(0.6923u_a + 0.4615v_a)$$

Example (3)

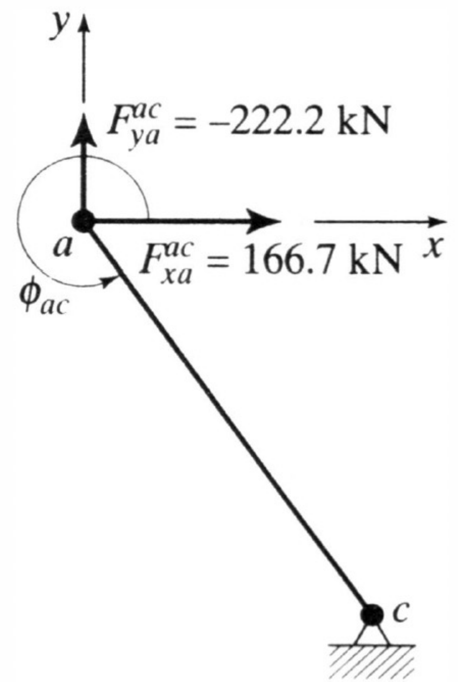
Consider *ac*:

$$\left(\frac{EA}{L}\right)_{ac} = \frac{200 \times 8 \times 10^3}{5 \times 10^3} = 320.0 \text{ kN/mm}$$

$$\phi_{ac} = \tan^{-1}\left(-\frac{4}{3}\right) = 306.87°$$

$$F_{xa}^{ac} = 320.0(\cos^2 \phi_{ac} \cdot u_a + \sin \phi_{ac} \cos \phi_{ac} \cdot v_a)$$

$$166.7 = 320.0(0.3600u_a - 0.4800v_a)$$



Example (4)

Solve Equations a and b simultaneously:

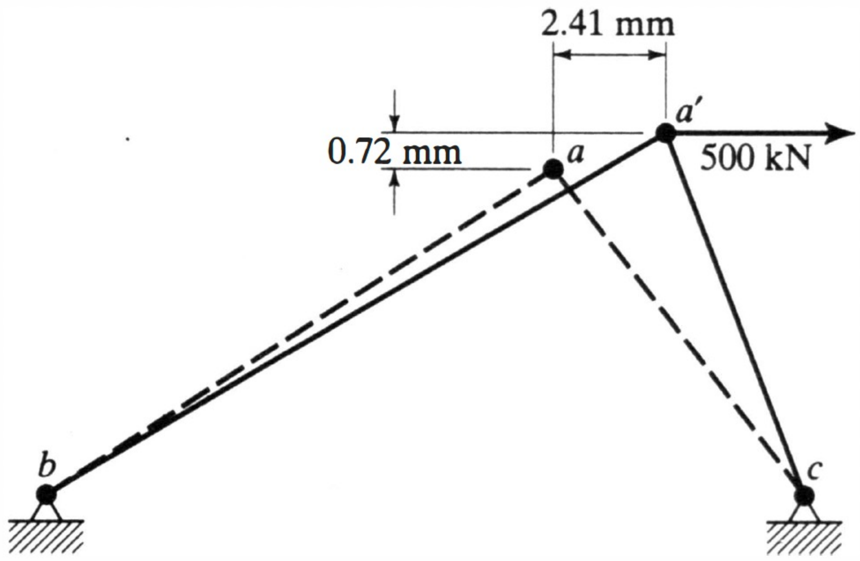
$$0.6923u_a + 0.4615v_a = 2.003 \quad \text{(a)}$$

$$0.3600u_a - 0.4800v_a = 0.5209 \quad \text{(b)}$$

$$u_a = 2.41 \text{ mm} \rightarrow$$

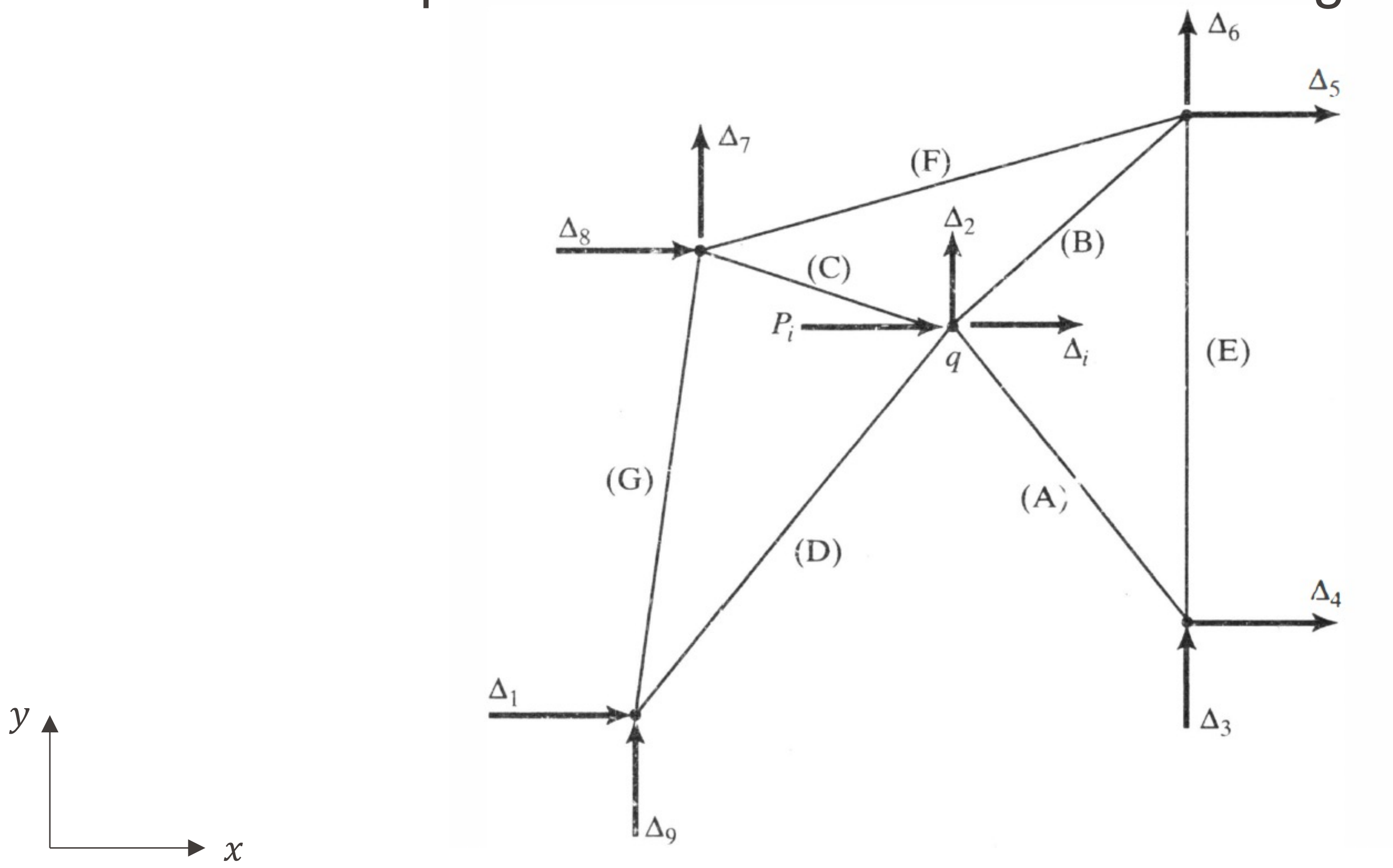
$$v_a = 0.72 \text{ mm} \uparrow$$

$$\overline{aa'} = 2.52 \text{ mm} \nearrow$$



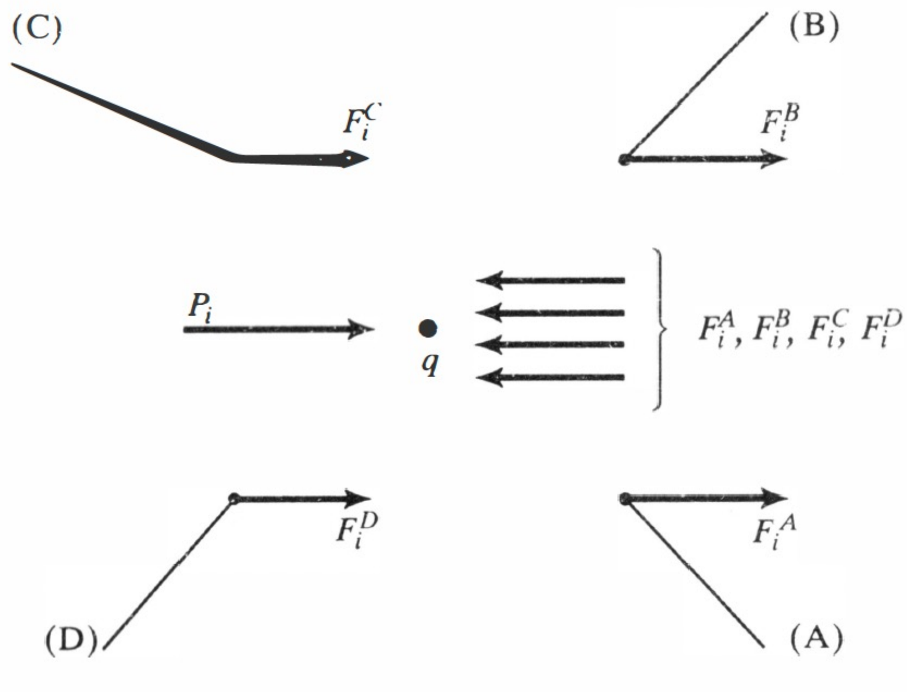
The direct stiffness method

Let us assume the plane truss below with identified degrees of freedom

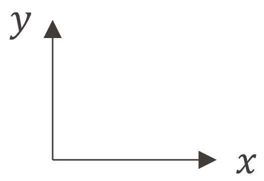


The direct stiffness method (2)

At junction q



Global x-direction internal force on bars



$$P_i = F_i^A + F_i^B + F_i^C + F_i^D$$

The direct stiffness method (3)

These forces yield expressions for $F_i^A, \dots F_i^D$ in terms of the corresponding element degrees of freedom $\Delta_i^A, \dots \Delta_9^D$; hence,

$$P_i = F_i^A + F_i^B + F_i^C + F_i^D =$$

$$\begin{aligned} P_i = {} & k_{ii}^A \Delta_i^A + k_{i2}^A \Delta_2^A + k_{i3}^A \Delta_3^A + k_{i4}^A \Delta_4^A \\ & + k_{ii}^B \Delta_i^B + k_{i2}^B \Delta_2^B + k_{i3}^B \Delta_3^B + k_{i4}^B \Delta_4^B \\ & + k_{ii}^C \Delta_i^C + k_{i2}^C \Delta_2^C + k_{i3}^C \Delta_3^C + k_{i4}^C \Delta_4^C \\ & + k_{ii}^D \Delta_i^D + k_{i2}^D \Delta_2^D + k_{i3}^D \Delta_3^D + k_{i4}^D \Delta_4^D \end{aligned}$$

The direct stiffness method (4)

At the joint the condition of displacement compatibility applies

$$\Delta_i^A = \Delta_i^B = \Delta_i^C = \Delta_i^D = \Delta_i$$

Therefore,

$$P_i = (k_{ii}^A + k_{ii}^B + k_{ii}^C + k_{ii}^D)\Delta_i + k_{i1}^D\Delta_1 \ (k_{i2}^A + k_{i2}^B + k_{i2}^C + k_{i2}^D)\Delta_2 + k_{i3}^A\Delta_3 + \ldots\ldots k_{i9}^D\,\Delta_9$$

or

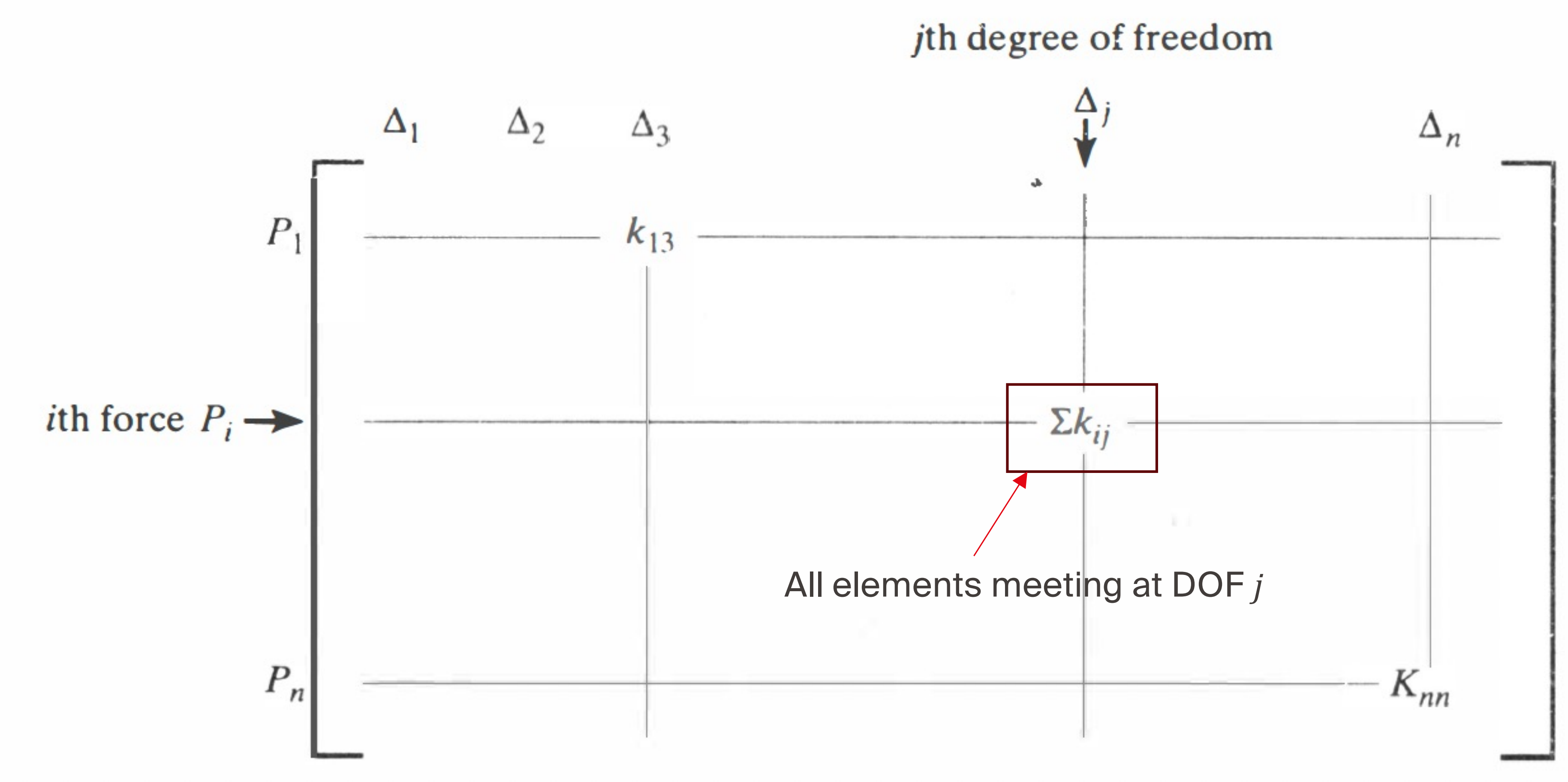
$$P_i = K_{ii}\Delta_i + K_{i1}\Delta_2 + K_{i3}\Delta_3 + \ldots\ldots K_{i9}\,\Delta_9$$

Global stiffness coefficients

The direct stiffness method (5)

- Each element stiffness coefficient is assigned a double subscript (k_{ij}). The first subscript in the element stiffness coefficient designates the force for which the equation is written, the second subscript designates the degree of freedom.
- Global stiffness matrix is aways square whose size is equal to the number of degrees of freedom in the complete system (see next page).
  - First subscript pertains to the force equation
  - Second subscript to the degree of freedom
- Support conditions are accounted for by noting which displacements are zero and then removing from the equations the columns or stiffness coefficients multiplying these degrees of freeedom.
- The remaining equations are solved to the global coordinate system.
- Member forces are then computed in the local coordinate system.

The direct stiffness method (6)



The direct stiffness method (7)

In a matrix form:

$$\{\mathbf{P}\} = [\mathbf{K}]\{\boldsymbol{\Delta}\}$$

Assume that the support degrees of freedom $\{\boldsymbol{\Delta}_s\}$ are grouped together

After reordering the equations,

Remaining degrees of freedom

$$\begin{Bmatrix} \mathbf{P}_f \\ \hline \mathbf{P}_s \end{Bmatrix} = \left[\begin{array}{c|c} \mathbf{K}_{ff} & \mathbf{K}_{fs} \\ \hline \mathbf{K}_{sf} & \mathbf{K}_{ss} \end{array}\right] \begin{Bmatrix} \boldsymbol{\Delta}_f \\ \hline \boldsymbol{\Delta}_s \end{Bmatrix}$$

supports

Note that for the supports:

$$\{\boldsymbol{\Delta}_s\} = 0$$

The direct stiffness method (8)

Therefore,

$$\{\mathbf{P}_f\} = [\mathbf{K}_{ff}]\{\mathbf{\Delta}_f\} \qquad\qquad \{\mathbf{P}_s\} = [\mathbf{K}_{sf}]\{\mathbf{\Delta}_f\}$$

To compute the displacements at all unsupported nodes:

$$\{\mathbf{\Delta}_f\} = [\mathbf{K}_{ff}]^{-1}\{\mathbf{P}_f\} = [\mathbf{D}]\{\mathbf{P}_f\}$$

Global flexibility matrix

To compute the support reactions:

$$\{\mathbf{P}_s\} = [\mathbf{K}_{sf}]\,[\mathbf{D}]\{\mathbf{P}_f\}$$

To obtain the internal force distribution in the i-th element

$$\{\mathbf{F}^i\} = [\mathbf{k}^i]\,\{\mathbf{\Delta}^i\}$$

Static condensation

The term condensation refers to the contraction in size of a system of equations by elimination of certain degrees of freedom.

$$\begin{Bmatrix} \mathbf{P}_c \\ \hline \mathbf{P}_b \end{Bmatrix} = \left[\begin{array}{c|c} \mathbf{K}_{cc} & \mathbf{K}_{cb} \\ \hline \mathbf{K}_{bc} & \mathbf{K}_{bb} \end{array}\right] \begin{Bmatrix} \boldsymbol{\Delta}_c \\ \hline \boldsymbol{\Delta}_b \end{Bmatrix}$$

And condenses them to the form:

$$\{\widehat{\mathbf{P}}_c\} = [\widehat{\mathbf{K}}_{cc}]\{\boldsymbol{\Delta}_c\}$$

First solve the lower partition and solve for $\{\boldsymbol{\Delta}_b\}$

$$\{\boldsymbol{\Delta}_b\} = [\mathbf{K}_{bb}]^{-1}\{\mathbf{P}_b\} - [\mathbf{K}_{bb}]^{-1}[\mathbf{K}_{bc}]\{\boldsymbol{\Delta}_c\}$$

Static condensation (2)

Substituting the previous equation into the expanded upper partition of the system:

$$\left\{\begin{array}{c}\mathbf{P}_c \\ \hline \mathbf{P}_b\end{array}\right\} = \left[\begin{array}{c|c}\mathbf{K}_{cc} & \mathbf{K}_{cb} \\ \hline \mathbf{K}_{bc} & \mathbf{K}_{bb}\end{array}\right]\left\{\begin{array}{c}\boldsymbol{\Delta}_c \\ \hline \boldsymbol{\Delta}_b\end{array}\right\}$$

$$\{\boldsymbol{P}_c\} = [\mathbf{K}_{cc}]\{\boldsymbol{\Delta}_c\} + [\mathbf{K}_{cb}]\,\{\boldsymbol{\Delta}_b\}$$

$$\{\boldsymbol{P}_c\} = [\mathbf{K}_{cc}]\{\boldsymbol{\Delta}_c\} - [\mathbf{K}_{cb}][\mathbf{K}_{bb}]^{-1}[\mathbf{K}_{bc}]\,\{\boldsymbol{\Delta}_c\} + [\mathbf{K}_{cb}][\mathbf{K}_{bb}]^{-1}\,\{\mathbf{P}_b\}$$

$$\underbrace{\{\boldsymbol{P}_c\} - [\mathbf{K}_{cb}][\mathbf{K}_{bb}]^{-1}\,\{\mathbf{P}_b\}}_{\{\widehat{\mathbf{P}}_c\}} = \underbrace{\left([\mathbf{K}_{cc}] - [\mathbf{K}_{cb}][\mathbf{K}_{bb}]^{-1}[\mathbf{K}_{bc}]\right)}_{[\widehat{\mathbf{K}}_{cc}]}\,\{\boldsymbol{\Delta}_c\}$$

Static condensation (3)

Therefore, the unknown displacements can be calculated as follows:

$$\{\boldsymbol{\Delta}_c\} = [\widehat{\mathbf{K}}_{cc}]^{-1}\{\widehat{\mathbf{P}}_c\}$$

And finally,

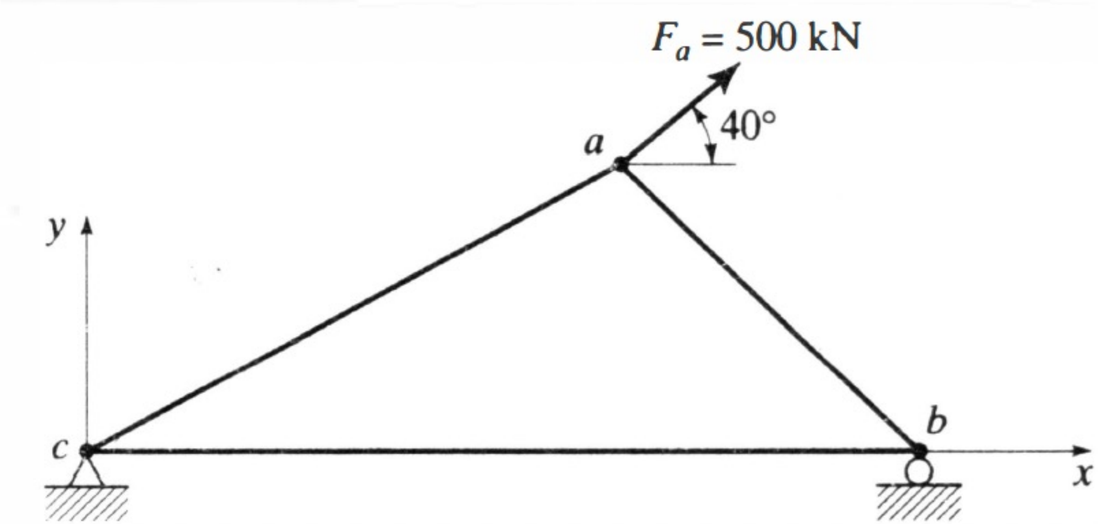
$$\{\boldsymbol{\Delta}_b\} = [\mathbf{K}_{bb}]^{-1}\,\{\mathbf{P}_b\} - [\mathbf{K}_{bb}]^{-1}[\mathbf{K}_{bc}]\,\{\boldsymbol{\Delta}_{\boldsymbol{c}}\}$$

The inversion of matrices $[\mathbf{K}_{bb}]$ and $[\widehat{\mathbf{K}}_{cc}]^{-1}$ are easier to handle than the original global stiffness matrix $[\mathbf{K}]$, which is usually ill-conditioned.

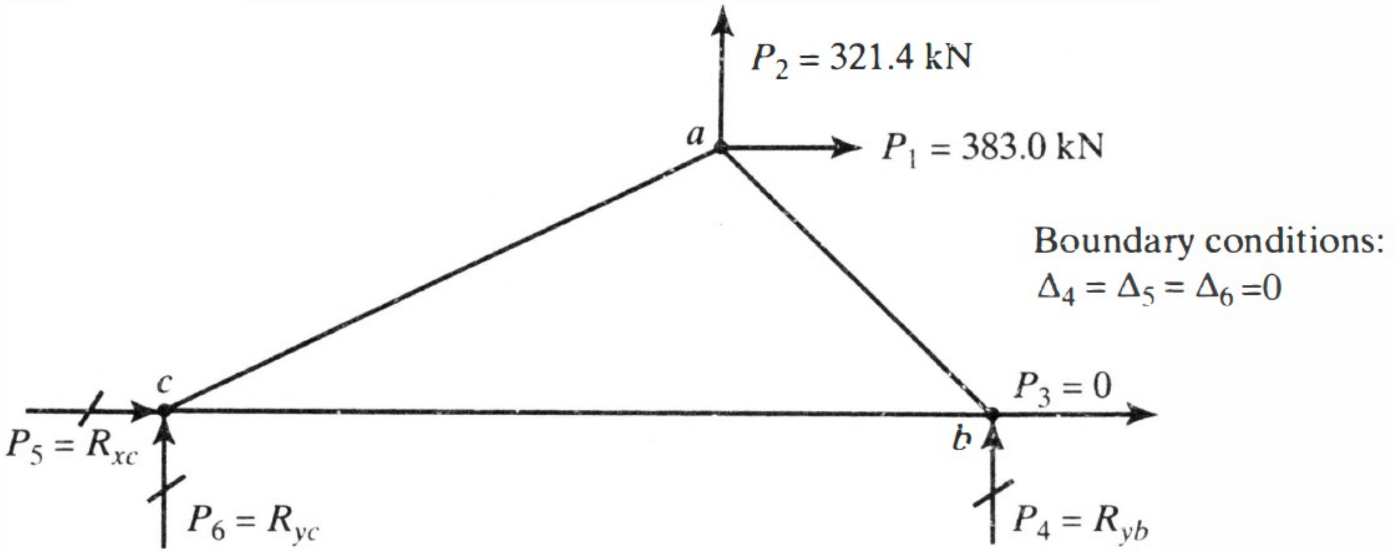
Example

The truss is supported and loaded as shown in the figure.

- Calculate the displacements at a and b.
- Calculate the reactions.
- Calculate the bar forces.



Example (2)



1. Displacements. The upper three global stiffness equations can be written as follows:

$$\begin{Bmatrix} 383.0 \\ 321.4 \\ 0 \end{Bmatrix} = 10^2 \begin{bmatrix} 6.348 & -1.912 & -3.536 \\ & 4.473 & 3.536 \\ \text{Sym.} & & 6.830 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} + 10^2 \begin{bmatrix} 3.536 & -2.812 & -1.624 \\ -3.536 & -1.624 & -0.938 \\ -3.536 & -3.294 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Example (3)

Inverting the first matrix and solving for the displacements yields

$$\lfloor \Delta_1 \quad \Delta_2 \quad \Delta_3 \rfloor = \lfloor 0.871 \quad 1.244 \quad -0.193 \rfloor \text{ mm}$$

Example (4)

3. Bar forces. The bar forces may now be obtained from the member stiffness equations:
Member *ab*

$$\begin{Bmatrix} F_1^{ab} \\ F_2^{ab} \end{Bmatrix} = 707.11 \begin{array}{c} \begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \end{array} \\ \begin{bmatrix} 0.500 & -0.500 & -0.500 \\ -0.500 & 0.500 & 0.500 \end{bmatrix} \end{array} \begin{Bmatrix} 0.871 \\ 1.244 \\ -0.193 \end{Bmatrix} = \begin{Bmatrix} -63.6 \\ 63.6 \end{Bmatrix} \text{ kN}$$

$$F_{ab} = F_2^{ab} \cdot \sqrt{2} = +90.0 \text{ kN (tension)}$$

Example (5)

Member *ac*

$$\begin{Bmatrix} F_1^{ac} \\ F_2^{ac} \end{Bmatrix} = 375.00 \begin{array}{cc} \Delta_1 & \Delta_2 \end{array} \begin{bmatrix} 0.750 & 0.433 \\ 0.433 & 0.250 \end{bmatrix} \begin{Bmatrix} 0.871 \\ 1.244 \end{Bmatrix} = \begin{Bmatrix} 447.0 \\ 258.0 \end{Bmatrix} \text{ kN}$$

$$F_{ac} = F_1^{ac} \cdot \frac{2}{\sqrt{3}} = +516.2 \text{ kN (tension)}$$