9 CHAPTER

CONSTRAINTS

Constraints enforce a relationship among d.o.f. Procedures for imposing a constraint include transformation, Lagrange multipliers, and penalty functions. Naturally arising constraints, constraint counting, and integration rules for incompressible materials are also discussed.

9.1 CONSTRAINTS. TRANSFORMATIONS

Constraints. A *constraint* either prescribes the value of a d.o.f. (as in imposing a support condition) or prescribes a relationship among d.o.f. In common terminology, a *single-point constraint* sets a single d.o.f. to a known value (often zero), and a *multipoint constraint* imposes a relationship between two or more d.o.f. Thus support conditions in the three-bar truss of Fig. 2.2-1 invoke three single-point constraints. Rigid links and rigid elements, discussed in Section 7.8, each invoke a multipoint constraint.

Figure 9.1-1 shows an example in which constraints could be imposed. In a typical frame, axial deformation of a member can usually be ignored; only bending deformation is significant. Accordingly, in Fig. 9.1-1, one could impose the single-point constraints $v_A = 0$ and $v_B = 0$, and the multipoint constraint $u_A = u_B$, after which the active d.o.f. consist of only θ_A, θ_B, and either u_A or u_B. (Failure to impose the constraint $u_A = u_B$ invites numerical difficulty; see Section 18.2.) Special-purpose computer programs for tall buildings may incorporate constraints of this type by allowing only three d.o.f. per floor, these being the rotation θ_z of a floor about a vertical z axis and the horizontal displacement components u and v.

For each equation of constraint, one d.o.f. can be eliminated from the vector of structural d.o.f. {D}. However, doing so may involve appreciable manipulation and typically increases the bandwidth (or the frontwidth) of the structural equations. The Lagrange multiplier method of treating constraints, discussed subsequently, *adds* to the number of equations but requires less manipulation.

Transformation Equations. Constraint equations that couple d.o.f. in {D} can be written in the form

$$[\mathbf{C}]\{\mathbf{D}\} = \{\mathbf{Q}\} \tag{9.1-1}$$

where [C] and {Q} contain constants. There are more d.o.f. in {D} than constraint equations, so [C] has more columns than rows. We now consider the common case {Q} = {0}. Let Eq. 9.1-1 be partitioned so that

$$[\mathbf{C}_r \quad \mathbf{C}_c]\begin{Bmatrix}\mathbf{D}_r\\ \mathbf{D}_c\end{Bmatrix} = \{0\} \tag{9.1-2}$$

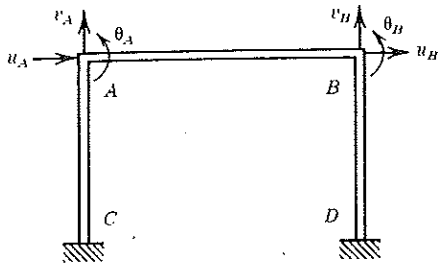


Figure 9.1-1. A three-element plane frame, fixed at nodes C and D. D.o.f. at nodes A and B are shown.

where $\{\mathbf{D}_r\}$ and $\{\mathbf{D}_c\}$ are, respectively, d.o.f. to be retained and d.o.f. to be eliminated or "condensed out." Because there are as many d.o.f. $\{\mathbf{D}_c\}$ as there are independent equations of constraint in Eq. 9.1-2, matrix $[\mathbf{C}_c]$ is square and nonsingular. Solution for $\{\mathbf{D}_c\}$ yields

$$\{\mathbf{D}_c\} = [\mathbf{C}_{rc}]\{\mathbf{D}_r\}, \qquad \text{where} \quad [\mathbf{C}_{rc}] = -[\mathbf{C}_c]^{-1}[\mathbf{C}_r] \tag{9.1-3}$$

We now write as one relation the identity $\{\mathbf{D}_r\} = \{\mathbf{D}_r\}$ and Eq. 9.1-3:

$$\begin{Bmatrix} \mathbf{D}_r \\ \mathbf{D}_c \end{Bmatrix} = [\mathbf{T}]\{\mathbf{D}_r\}, \qquad \text{where} \quad [\mathbf{T}] = \begin{bmatrix} \mathbf{I} \\ \mathbf{C}_{rc} \end{bmatrix} \tag{9.1-4}$$

With the transformation matrix $[\mathbf{T}]$ now defined, the familiar transformations $\{\mathbf{R}\} = [\mathbf{T}]^T\{\mathbf{R}'\}$ and $[\mathbf{K}] = [\mathbf{T}]^T[\mathbf{K}'][\mathbf{T}]$ of Eqs. 7.4-2 and 7.4-4 can be applied to the structural equations $[\mathbf{K}']\{\mathbf{D}'\} = \{\mathbf{R}'\}$, which are partitioned as

$$\begin{bmatrix} \mathbf{K}_{rr} & \mathbf{K}_{rc} \\ \mathbf{K}_{cr} & \mathbf{K}_{cc} \end{bmatrix} \begin{Bmatrix} \mathbf{D}_r \\ \mathbf{D}_c \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_r \\ \mathbf{R}_c \end{Bmatrix} \tag{9.1-5}$$

The condensed system is

$$[\mathbf{K}_{rr} + \mathbf{K}_{rc}\mathbf{C}_{rc} + \mathbf{C}_{rc}^T\mathbf{K}_{cr} + \mathbf{C}_{rc}^T\mathbf{K}_{cc}\mathbf{C}_{rc}]\{\mathbf{D}_r\} = \{\mathbf{R}_r + \mathbf{C}_{rc}^T\mathbf{R}_c\} \tag{9.1-6}$$

After Eq. 9.1-6 is solved for $\{\mathbf{D}_r\}$, Eq. 9.1-3 yields $\{\mathbf{D}_c\}$. If $\{\mathbf{Q}\} \neq \{\mathbf{0}\}$ in Eq. 9.1-1, additional terms appear on the right-hand side of Eq. 9.1-6.

If Eq. 9.1-2 simply sets certain d.o.f. $\{\mathbf{D}_c\}$ to zero, then $[\mathbf{C}_r] = [\mathbf{0}]$ and $[\mathbf{C}_c] = [\mathbf{I}]$, hence $[\mathbf{C}_{rc}] = [\mathbf{0}]$, and Eq. 9.1-6 is equivalent to discarding rows and columns associated with $\{\mathbf{D}_c\}$. Otherwise, the choice of which d.o.f. to place in $\{\mathbf{D}_c\}$ is not unique, so the choice of $[\mathbf{C}_c]$ is not unique. One might then define $[\mathbf{C}_c]$ to be the last c linearly independent columns of $[\mathbf{C}]$.

It is possible to avoid the reordering, partitioning, and matrix multiplications implied by Eq. 9.1-6 by applying individual constraint equations serially and retaining all d.o.f. of $\{\mathbf{D}_r\}$ and $\{\mathbf{D}_c\}$ in the transformed equations [6.1]. The transformed coefficient matrix may not be positive definite.

Example. Consider the three-element structure of Fig. 9.1-2. With only axial deformation allowed, and after the support condition $u = 0$ is imposed at $x = 0$, the structural equations are

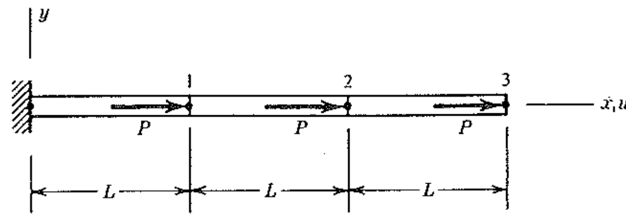


Figure 9.1-2. Three identical bar elements, each of axial stiffness $k = AE/L$.

$$\begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ P \\ P \end{Bmatrix} \tag{9.1-7}$$

Imagine that the constraint $u_2 = u_3$ is to be imposed. With the choice $D_c = u_3$, Eqs. 9.1-2 and 9.1-3 become

$$[0 \quad 1 \mid -1] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0 \qquad \text{and} \qquad [\mathbf{C}_{rc}] = [0 \quad 1] \tag{9.1-8}$$

The transformation matrix of Eq. 9.1-4 and the reduced system of Eq. 9.1-6 are

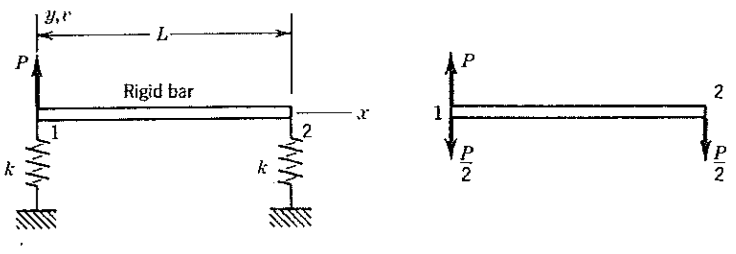
$$[\mathbf{T}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 2P \end{Bmatrix} \tag{9.1-9}$$

Equation 9.1-9 yields $u_1 = 3P/k$ and $u_2 = 5P/k$. Hence, Eq. 9.1-8 yields $u_3 = 5P/k$.

In Section 8.16, we note that two different nodes can be forced to have the same d.o.f. in $\{\mathbf{D}\}$ by giving them the same node number. (Actual nodal coordinates are still used in the generation of element matrices.) Thus a node whose d.o.f. would all appear in $\{\mathbf{D}_c\}$ can be assigned a node number associated with $\{\mathbf{D}_r\}$ instead of using the transformation, Eq. 9.1-6. Any externally applied loads on d.o.f. $\{\mathbf{D}_c\}$ must be transferred to d.o.f. in $\{\mathbf{D}_r\}$. In applying this method to the foregoing example problem, one assigns the number 2 to the rightmost two nodes. This causes addition of the four coefficients in $[\mathbf{k}]$ of the right element, for a sum of zero at node 3, effectively removing the right element (but not its load) from the structure, and producing Eq. 9.1-9 upon assembly of the remaining two elements.

The condensed system in Eq. 9.1-6 is different from the system obtained by static condensation, Eq. 8.1-3. In Eq. 8.1-3, condensed d.o.f. are related to retained d.o.f. by equilibrium equations already present in the system $[\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{R}\}$. In Eq. 9.1-6, condensed d.o.f. $\{\mathbf{D}_c\}$ are related to retained d.o.f. $\{\mathbf{D}_r\}$ by *supplementary* equations of constraint that replace certain equilibrium equations. Accordingly, constraints may appear to falsify certain equilibrium equations. Figure 9.1-3 is a case in point. The original system, and the system that results from the constraint $v_1 = v_2$, are respectively

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix} \qquad \text{and} \qquad (2k)v_1 = P \tag{9.1-10}$$



(*a*) (*b*)

Figure 9.1-3. (*a*) A rigid bar supported by two springs. (*b*) External and elastic forces applied to the bar if the constraint $v_1 = v_2$ is imposed. Forces of constraint are not shown.

Hence, $v_1 = v_2 = P/2k$, and forces carried by the springs are $kv_1 = kv_2 = P/2$. Net forces applied to the bar, Fig. 9.1-3*b*, satisfy equilibrium of *y*-direction forces but not moment equilibrium. Of course, the condensed structure is not that of Fig. 9.1-3*b*; it is a single spring of stiffness $2k$, loaded by force *P*.

9.2 LAGRANGE MULTIPLIERS

Lagrange's method of undetermined multipliers is used to find the maximum or minimum of a function whose variables are not independent but have some prescribed relation. In structural mechanics the function is potential energy Π_p and the variables are d.o.f. in $\{\mathbf{D}\}$. System unknowns become $\{\mathbf{D}\}$ *and* the Lagrange multipliers.

The theory is easy to describe. We write the constraint equation (Eq. 9.1-1) as the homogeneous equation $[\mathbf{C}]\{\mathbf{D}\} - \{\mathbf{Q}\} = \{\mathbf{0}\}$ and multiply its left-hand side by a row vector $\{\boldsymbol{\lambda}\}^T$ that contains as many Lagrange multipliers λ_i as there are constraint equations. Next we add the result to the potential expression, Eq. 4.1-7:

$$\Pi_p = \tfrac{1}{2}\{\mathbf{D}\}^T[\mathbf{K}]\{\mathbf{D}\} - \{\mathbf{D}\}^T\{\mathbf{R}\} + \{\boldsymbol{\lambda}\}^T([\mathbf{C}]\{\mathbf{D}\} - \{\mathbf{Q}\}) \tag{9.2-1}$$

The expression in parentheses is zero, so we have added nothing to Π_p. Next we make Π_p stationary by writing the equations $\{\partial\Pi_p/\partial\mathbf{D}\} = \{\mathbf{0}\}$ and $\{\partial\Pi_p/\partial\boldsymbol{\lambda}\} = \{\mathbf{0}\}$, following differentiation rules stated in Appendix A. The result is

$$\begin{bmatrix} \mathbf{K} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{D} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R} \\ \mathbf{Q} \end{Bmatrix} \tag{9.2-2}$$

The lower partition of Eqs. 9.2-2 is Eq. 9.1-1, the equation of constraint. Equations 9.2-2 are solved for both $\{\mathbf{D}\}$ and $\{\boldsymbol{\lambda}\}$. The λ_i may be interpreted as forces of constraint (see the following example problem).

Strict partitioning—that is, $\{\mathbf{D}\}$ followed by $\{\boldsymbol{\lambda}\}$ in Eq. 9.2-2—increases bandwidth to the maximum. If instead the D_i and λ_i are interlaced, bandwidth can be much less, although not as small as when the λ_i are absent. However, in a Gauss elimination solution with pivoting on the diagonal, a zero pivot appears if a constraint equation is processed before any of the d.o.f. to which it is coupled. Otherwise, the null submatrix fills in and the solution proceeds normally if the stiffness matrix $[\mathbf{K}]$ is by itself positive definite.