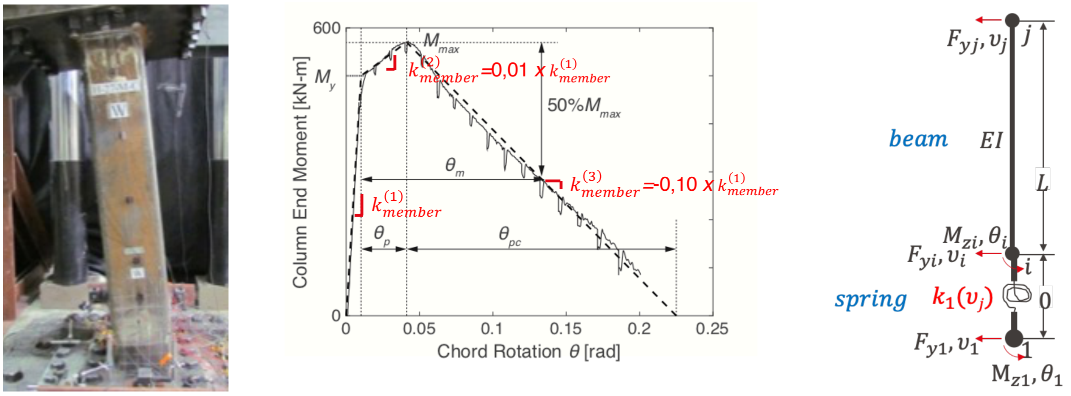
Question 1 (20 points)

Figure 1a shows a cantilever member with length, L, moment of inertia, I, and a Young's modulus, E. Its lateral response is shown in Figure 1b. We would like to approximate the cantilever member with a zero-length rotational element and an elastic beam-column element as shown in Figure 1c. The zero-length element is assumed to be infinitely stiff compared to the elastic beam-column element ("n" times stiffer than the corresponding beam).

Answer the following questions by showing all your derivations:

1. Express the rotational stiffness of the spring, $k_1^{(1)}$ as a function of k, E, I and L.
2. Express the post-yield hardening ratio, a_s, of the spring as a function of n.
3. Express the post-capping hardening ratio, a_c, of the spring as a function of n.



(a) Cantilever member (b) lateral response (c) model idealization

Figure 1. Cantilever member and model idealization

Solution

In all cases,

$$k_{member} = k_{numerical\ model} = \frac{1}{\frac{1}{k_1^i} + \frac{1}{k_{beam}}}$$

- If the elastic beam is rotationally rigid then, k_{beam} ~ infinite
  - Elastic range: $k_1^{(1)} = 3EI/L$
  - Post yield range : $k_1^{(2)} = 0{,}01 \times k_1^{(1)}$
  - Post capping range: $k_1^{(3)} = -\ 0{,}10 \times k_1^{(1)}$

$$k_1^{(1)} = nk_{beam} \quad (infinitely\ stiff\ in\ the\ elastic\ range, you\ can\ assume, n = 10)$$

$$k_{member} = \frac{1}{\frac{1}{k_1^{(1)}} + \frac{1}{k_{beam}}} = \frac{n \cdot k_{beam}^2}{(1+n) \cdot k_{beam}} \Rightarrow$$

$$k_{beam} = \frac{1+n}{n} \cdot k_{member} = \frac{1+n}{n} \cdot \frac{3EI}{L}$$

Hence,

$$k_1^{(1)} = (1+n) \cdot k_{member} = (1+n) \cdot \frac{3EI}{L}$$

$$k_{member}^{(2)} = \frac{1}{\frac{1}{k_1^{(2)}} + \frac{1}{k_{beam}}} = \frac{a_s \cdot k_1^{(1)} \cdot k_{beam}}{a_s \cdot k_1^{(1)} + k_{beam}}$$

$(assume\ k_1^{(2)} = a_s k_1^{(1)})$

$$k_{member}^{(2)} = \frac{a_s \cdot (1+n) \cdot k_{member} \cdot \frac{1+n}{n} \cdot k_{member}}{a_s \cdot (1+n) \cdot k_{member} + \frac{1+n}{n} \cdot k_{member}}$$

$$0{,}01 \cdot k_{member} = k_{member}^{(2)} = \frac{a_s \cdot (1+n) \cdot \frac{1+n}{n} \cdot k_{member}}{a_s \cdot (1+n) + \frac{1+n}{n}}$$

$$0{,}01 = \frac{a_s \cdot (1+n) \cdot \frac{1+n}{n}}{a_s \cdot (1+n) + \frac{1+n}{n}}$$

$$0{,}01 a_s (1+n) n - a_s (1+n)^2 = -0{,}03(1+n)$$

$$a_s = \frac{-0{,}01}{0{,}03n - n - 1}$$

and

$$a_{s,post-capping} = \frac{0.10}{-0{,}10n - n - 1}$$

Question 2 (50 points + 10points bonus)

Part A (10 points)

1. Provide at least three shortcomings of displacement-based beam-column elements.
2. Explain why the state determination of force-based beam-column elements is challenging compared to displacement-based elements?

Part B (40 points + 10 points bonus)

The tapered beam shown in Figure 2 has a linear elastic material. The material modulus of elasticity E is constant. The beam depth changes linearly from 2d at the fixed support to d at the tip. The beam width, b, is constant. This beam is analyzed with a single displacement-based beam-column element with two nodes. The left note is *Node-i* and the right node is *Node-j* as shown in the figure.

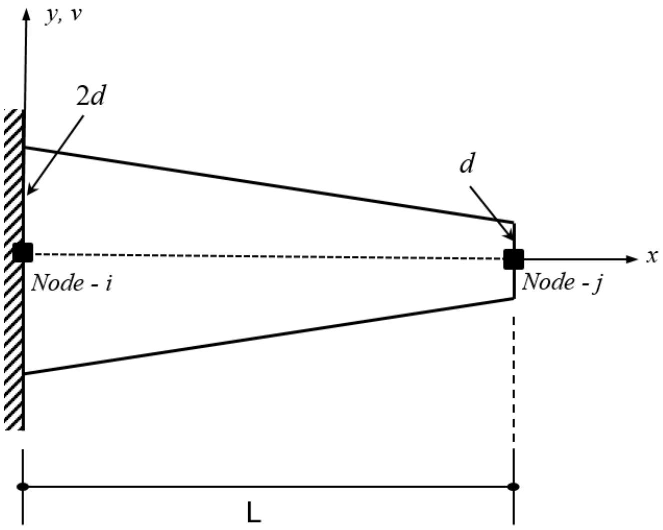


Figure 2. tapered element

The transverse displacement field $v(x)$ along the beam is approximated by the Euler-Bernoulli beam theory assumptions and,

$$v(x) = \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] v_j + \left[\frac{x^3}{L^2} - \frac{x^2}{L}\right] \theta_j \tag{1}$$

From the above displacement field, the curvature field $k(x)$ can be calculated to get the following transformation matrix,

$$k(x) = \mathbf{B(x)} \cdot \begin{Bmatrix} v_j \\ \theta_j \end{Bmatrix} \tag{2}$$

With the use of the principle of virtual displacement method, the resulting stiffness matrix of the element is 2x2 and can be calculated as:

$$\mathbf{k} = \int_0^L [\mathbf{B(x)}]^T \mathbf{k}_s(x)\, [\mathbf{B(x)}] dx \tag{3}$$

Where $\mathbf{k}_s(x) = EI(x)$ is the section stiffness matrix.

Answer to the following questions:

1. (10 points) What type of numerical integration method do you propose in order to calculate the above stiffness matrix “numerically exact”? How many integration points should be used with this method and explain why?
2. (20 points) Calculate the “numerically exact” stiffness matrix.
3. (10 points) Is this stiffness matrix the “theoretically exact” stiffness matrix for this tapered beam? Explain why?
4. **Bonus:** Calculate the “theoretically exact” stiffness matrix (10 **extra points**)

Solution

Part A
Question 1:

- The assumed cubic interpolation functions result in linear curvature and constant axial strain along the member.
- Have well-known issues when it comes to softening
- Generally require a large number of element segments (fine discretization mesh in the inelastic regions) to trace accurately the strain gradient along a member.
- A fine mesh (i.e., more element segments to discretize a member) increase the number of degrees-of-freedom of the problem thereby causing increased computational cost.

Question 2:
The state determination computation is challenging because (a) the flexibility (not stiffness) matrix and (b) the deformation vector that corresponds to the applied forces should be computed.

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Part B

1. What type of numerical integration method do you propose in order to calculate the above stiffness matrix “numerically exact”? How many integration points should be used with this method and explain why?

Since this is a problem with linear elasticity I propose to use Gauss-Legendre (herein just Gauss) quadrature because this method gives the best accuracy for the number of points required. The number of integration points is to be determined based on the order of the polynomial function to be integrated.

First the transverse displacement is defined,

```
In[2]:= v = {3 * (x / l) ^2 - 2 (x / l) ^3, x^3 / l^2 - x^2 / l};
```

The second derivative of the displacement field is required to obtain the strain-displacement matrix for flexure, B (strainDispMat),

```
In[3]:= strainDispMat = D[v, {x, 2}];
        strainDispMat // MatrixForm
```

Out[4]//MatrixForm=

$$\begin{pmatrix} \frac{6}{l^2} - \frac{12\,x}{l^3} \\ -\frac{2}{l} + \frac{6\,x}{l^2} \end{pmatrix}$$

Next, I define the moment of inertia as a function of x (ix),

```
In[5]:= ix = b / 12 * (2 * d - d * x / l) ^3 // FullSimplify
```

Out[5]= $\frac{1}{12}\, b\, d^3 \left(2 - \frac{x}{l}\right)^3$

The term b d^3/12 is defined as I_0 (i0) that represents the moment of inertia at the fixed end, and the substitution is made

```
In[6]:= ix = ix /. (b * d^3 / 12 → i0)
```

Out[6]= $i0 \left(2 - \frac{x}{l}\right)^3$

The stiffness matrix to be computed is

$k = \int_0^L B^T\, k_s\, B\, \mathrm{dx} = \int_0^L f[x]\, \mathrm{dx}$,

Since we are using numerical integration I define the integrand f[x] as a function of x. This is a 2x2 matrix function.

```
In[7]:= f[x_] = ee * ix * Outer[Times, strainDispMat, strainDispMat];
        f[x] // MatrixForm
```

Out[8]//MatrixForm=

$$\begin{pmatrix} ee\, i0 \left(\frac{6}{l^2} - \frac{12\,x}{l^3}\right)^2 \left(2 - \frac{x}{l}\right)^3 & ee\, i0 \left(\frac{6}{l^2} - \frac{12\,x}{l^3}\right) \left(-\frac{2}{l} + \frac{6\,x}{l^2}\right) \left(2 - \frac{x}{l}\right)^3 \\ ee\, i0 \left(\frac{6}{l^2} - \frac{12\,x}{l^3}\right) \left(-\frac{2}{l} + \frac{6\,x}{l^2}\right) \left(2 - \frac{x}{l}\right)^3 & ee\, i0 \left(-\frac{2}{l} + \frac{6\,x}{l^2}\right)^2 \left(2 - \frac{x}{l}\right)^3 \end{pmatrix}$$

Note that the highest power of x in f[x] is x^5. Gauss quadrature integrates polynomials exactly to the order of 2m-1, therefore 3-point integration is required.

2. Calculate the “numerically exact” stiffness matrix.

The integration will now be done with Gauss quadrature. The three Gauss points and associated weights are:
r1 = 0, w1 = 8/9
r2 = -Sqrt[3/5], w2 = 5/9
r3 = Sqrt[3/5], w3 = 5/9.

We also need the conversion between the natural coordinates r, and the given coordinates x:

In[9]:=
```
r1 = 0; w1 = 8. / 9.;
r2 = -Sqrt[3 / 5.]; w2 = 5. / 9;
r3 = Sqrt[3 / 5.]; w3 = 5. / 9;
r[x_] = l / 2 + x * l / 2;
j = D[r[x], x];
```

I define three functions for each of the entries of the matrix function f[x], note that f_12 = f_21 due to symmetry of the stiffness matrix:

In[14]:=
```
f11[x_] = f[x][[1, 1]]
f12[x_] = f[x][[1, 2]]
f22[x_] = f[x][[2, 2]]
```

Out[14]= $ee\ i0 \left(\frac{6}{l^2} - \frac{12\,x}{l^3}\right)^2 \left(2 - \frac{x}{l}\right)^3$

Out[15]= $ee\ i0 \left(\frac{6}{l^2} - \frac{12\,x}{l^3}\right) \left(-\frac{2}{l} + \frac{6\,x}{l^2}\right) \left(2 - \frac{x}{l}\right)^3$

Out[16]= $ee\ i0 \left(-\frac{2}{l} + \frac{6\,x}{l^2}\right)^2 \left(2 - \frac{x}{l}\right)^3$

Integration point 1 (r1 = 0):

In[17]:= `f11pt1 = f11[r[r1]]`

Out[17]= 0

In[18]:= `f12pt1 = f12[r[r1]]`

Out[18]= 0

In[19]:= `f22pt1 = f22[r[r1]] // N`

Out[19]= $\frac{3.375\ ee\ i0}{l^2}$

Integration point 2 (r2 = -Sqrt[3/5]):

In[20]:= `f11pt2 = f11[r[r2]]`

Out[20]= $\frac{145.203\ ee\ i0}{l^4}$

In[21]:= `f12pt2 = f12[r[r2]]`

Out[21]= $-\frac{41.3588\ ee\ i0}{l^3}$

In[22]:= `f22pt2 = f22[r[r2]]`

Out[22]= $\frac{11.7804\ ee\ i0}{l^2}$

Integration point 3 (r3 = -Sqrt[3/5]):

In[23]:= `f11pt3 = f11[r[r3]]`

Out[23]= $\frac{29.7571\ ee\ i0}{l^4}$

In[24]:= `f12pt3 = f12[r[r3]]`

Out[24]= $-\frac{21.2812\ ee\ i0}{l^3}$

In[25]:= `f22pt3 = f22[r[r3]]`

Out[25]= $\frac{15.2196\ ee\ i0}{l^2}$

Weighted sum

The integral is computed using Gauss quadrature

$\int_0^L f[x]\,\mathrm{dx} = \sum w_i\, f[r_i]\, J$

In[26]:=
```
k = (w1 * {{f11pt1, f12pt1}, {f12pt1, f22pt1}} + w2 * {{f11pt2, f12pt2}, {f12pt2, f22pt2}} +
     w3 * {{f11pt3, f12pt3}, {f12pt3, f22pt3}}) * j;
k // Chop // MatrixForm
```

Out[27]//MatrixForm=

$$\begin{pmatrix} \frac{48.6\ \text{ee i0}}{l^3} & -\frac{17.4\ \text{ee i0}}{l^2} \\ -\frac{17.4\ \text{ee i0}}{l^2} & \frac{9.\ \text{ee i0}}{l} \end{pmatrix}$$

The numerically exact stiffness matrix is:

$$\begin{pmatrix} \frac{48.6\, \mathrm{EI}_0}{L^3} & \frac{-17.4\, \mathrm{EI}_0}{L^2} \\ \frac{-17.4\, \mathrm{EI}_0}{L^2} & \frac{9\, \mathrm{EI}_0}{L} \end{pmatrix}$$

3. Is this stiffness matrix the “theoretically exact” stiffness matrix for this tapered beam? Explain why or why not?

The stiffness matrix computed using Gauss quadrature is the “theoretically exact” stiffness matrix disregarding any round-off errors introduced by finite-precision computations. This is because of our choice of 3-point Gaussian quadrature as explained earlier.

4. Bonus: Calculate the “theoretically exact” stiffness matrix (** extra points)

The “theoretically exact” stiffness matrix is found by evaluating the integral of f[x] analytically.

Evaluating each of the terms of f[x] individually to show potential steps,

The 1,1 entry

In[28]:= `Integrate[f11[x], x]`

Out[28]=
$$\frac{36\ \text{ee i0}\left(8\, l^5 x - 22\, l^4 x^2 + \frac{86\, l^3 x^3}{3} - \frac{73\, l^2 x^4}{4} + \frac{28\, l\, x^5}{5} - \frac{2\, x^6}{3}\right)}{l^9}$$

In[29]:= `Integrate[f11[x], {x, 0, l}]`

Out[29]=
$$\frac{243\ \text{ee i0}}{5\, l^3}$$

The 1,2 entry

In[30]:= `Integrate[f12[x], x]`

Out[30]=
$$-\frac{12\ \text{ee i0}\left(8\, l^5 x - 26\, l^4 x^2 + 38\, l^3 x^3 - \frac{103\, l^2 x^4}{4} + \frac{41\, l\, x^5}{5} - x^6\right)}{l^8}$$

In[31]:= `Integrate[f12[x], {x, 0, l}]`

Out[31]=
$$-\frac{87\ \text{ee i0}}{5\, l^2}$$

The 2,2 entry

In[32]:= `Integrate[f22[x], x]`

Out[32]=
$$\frac{4\ \text{ee i0}\left(8\, l^5 x - 30\, l^4 x^2 + 50\, l^3 x^3 - \frac{145\, l^2 x^4}{4} + 12\, l\, x^5 - \frac{3\, x^6}{2}\right)}{l^7}$$

In[33]:= `Integrate[f22[x], {x, 0, l}]`

Out[33]=
$$\frac{9\ \text{ee i0}}{l}$$

Complete matrix

In[34]:=
```
kTheory = Integrate[f[x], {x, 0, l}] // FullSimplify;
kTheory // MatrixForm
```

Out[35]//MatrixForm=

$$\begin{pmatrix} \frac{243\,ee\,i0}{5\,l^3} & -\frac{87\,ee\,i0}{5\,l^2} \\ -\frac{87\,ee\,i0}{5\,l^2} & \frac{9\,ee\,i0}{l} \end{pmatrix}$$

Evaluating this matrix numerically yields

In[36]:=
```
N[kTheory] // MatrixForm
```

Out[36]//MatrixForm=

$$\begin{pmatrix} \frac{48.6\,ee\,i0}{l^3} & -\frac{17.4\,ee\,i0}{l^2} \\ -\frac{17.4\,ee\,i0}{l^2} & \frac{9.\,ee\,i0}{l} \end{pmatrix}$$

As expected, the numerical and analytical stiffness matrices correspond.

CIVIL 449 Final Exam Problem 2 Solutions

In[1]:=
```
Clear["Global`*"]
```

1. What type of numerical integration method do you propose in order to calculate the above stiffness matrix “numerically exact”? How many integration points should be used with this method and explain why?

Since this is a problem with linear elasticity I propose to use Gauss-Legendre (herein just Gauss) quadrature because this method gives the best accuracy for the number of points required. The number of integration points is to be determined based on the order of the polynomial function to be integrated.

First the transverse displacement is defined,

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Out[5]= $\frac{1}{12}\,b\,d^3\,\left(2 - \frac{x}{l}\right)^3$

The term b d^3/12 is defined as I_0 (i0) that represents the moment of inertia at the fixed end, and the substitution is made

In[6]:=
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ix = ix /. (b * d ^ 3 / 12 → i0)
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Out[6]= $i0\,\left(2 - \frac{x}{l}\right)^3$

The stiffness matrix to be computed is

$k = \int_0^L B^T\,k_s\,B\,\mathrm{dx} \;=\; \int_0^L f[x]\,\mathrm{dx},$

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Note that the highest power of x in f[x] is x^5. Gauss quadrature integrates polynomials exactly to the order of 2m-1, therefore 3-point integration is required.

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The integration will now be done with Gauss quadrature. The three Gauss points and associated weights are:
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    w3 * {{f11pt3, f12pt3}, {f12pt3, f22pt3}}) * j;
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The numerically exact stiffness matrix is:

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3. Is this stiffness matrix the “theoretically exact” stiffness matrix for this tapered beam? Explain why or why not?

The stiffness matrix computed using Gauss quadrature is the “theoretically exact” stiffness matrix disregarding any round-off errors introduced by finite-precision computations. This is because of our choice of 3-point Gaussian quadrature as explained earlier.

4. Bonus: Calculate the “theoretically exact” stiffness matrix (** extra points)

The “theoretically exact” stiffness matrix is found by evaluating the integral of f[x] analytically.

Evaluating each of the terms of f[x] individually to show potential steps,

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Out[28]=
$$\frac{36\ \mathrm{ee\ i0}\left(8\, l^5 x - 22\, l^4 x^2 + \frac{86\, l^3 x^3}{3} - \frac{73\, l^2 x^4}{4} + \frac{28\, l\, x^5}{5} - \frac{2\, x^6}{3}\right)}{l^9}$$

In[29]:=
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Integrate[f11[x], {x, 0, l}]
```

Out[29]=
$$\frac{243\ \mathrm{ee\ i0}}{5\, l^3}$$

The 1,2 entry

In[30]:=
```
Integrate[f12[x], x]
```

Out[30]=
$$-\frac{12\ \mathrm{ee\ i0}\left(8\, l^5 x - 26\, l^4 x^2 + 38\, l^3 x^3 - \frac{103\, l^2 x^4}{4} + \frac{41\, l\, x^5}{5} - x^6\right)}{l^8}$$

In[31]:=
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Integrate[f12[x], {x, 0, l}]
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The 2,2 entry

In[32]:=
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Integrate[f22[x], x]
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Out[32]=
$$\frac{4\ \mathrm{ee\ i0}\left(8\, l^5 x - 30\, l^4 x^2 + 50\, l^3 x^3 - \frac{145\, l^2 x^4}{4} + 12\, l\, x^5 - \frac{3\, x^6}{2}\right)}{l^7}$$

In[33]:=
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Integrate[f22[x], {x, 0, l}]
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Out[33]=
$$\frac{9\ \mathrm{ee\ i0}}{l}$$

Complete matrix

In[34]:=
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kTheory = Integrate[f[x], {x, 0, l}] // FullSimplify;
kTheory // MatrixForm
```

Out[35]//MatrixForm=

$$\begin{pmatrix} \frac{243\ ee\ i0}{5\ l^3} & -\frac{87\ ee\ i0}{5\ l^2} \\ -\frac{87\ ee\ i0}{5\ l^2} & \frac{9\ ee\ i0}{l} \end{pmatrix}$$

Evaluating this matrix numerically yields

In[36]:=
```
N[kTheory] // MatrixForm
```

Out[36]//MatrixForm=

$$\begin{pmatrix} \frac{48.6\ ee\ i0}{l^3} & -\frac{17.4\ ee\ i0}{l^2} \\ -\frac{17.4\ ee\ i0}{l^2} & \frac{9.\ ee\ i0}{l} \end{pmatrix}$$

As expected, the numerical and analytical stiffness matrices correspond.

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Solution Q3

a) $①_1$ • Bar 1 is the bar that yields } either stated explicitly or implicitly (through assumptions in calculations)

$①_2$ All other bars remain elastic

• Force – displacement relationship of bar 1

assume: $A = 0.01\ m^2$

F [MN]	u_2 [m]	
0	0	
$A \cdot f_{y_2} = 2.0$ $①_3$	$\varepsilon \cdot l_1 = 0.002 \cdot 0.1 = 0.0002$	$①_4$
$A \cdot f_{y_2} = 2.0$	$\varepsilon \cdot l_1 = 0.006 \cdot 0.1 = 0.0006$	$①_5$
0	$\varepsilon \cdot l_1 = 0.008 \cdot 0.1 = 0.0008$	$①_6$

• Elastic deformation of bar 2 – 4 $\Delta = \varepsilon \cdot (l_2 + l_3 + l_4) = \frac{200}{240} \cdot 0.002 \cdot 0.3$

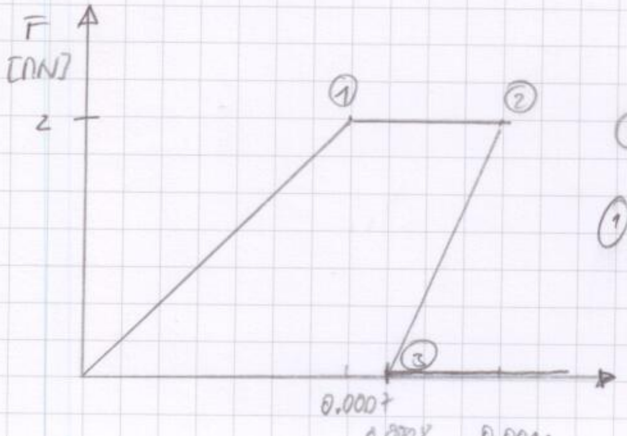
$①_7$ – idea to calculate

$= 0.0005$ [m]

$①_8$ value correct

• F [MN] $u_5 = u_2 + \Delta$ [m]

	F	u_5	
	0	$0 + 0 = 0$	
$①_9$	2	$0.0002 + 0.0005 = 0.0007$	$①_{12}$
$①_{10}$	2	$0.0006 + 0.0005 = 0.0011$	$①_{13}$
$①_{11}$	0	$0.0008 + 0 = 0.0008$	$①_{14}$



$①_{15}$ Sketch correct up to point ②

$①_{16}$ up to point ③

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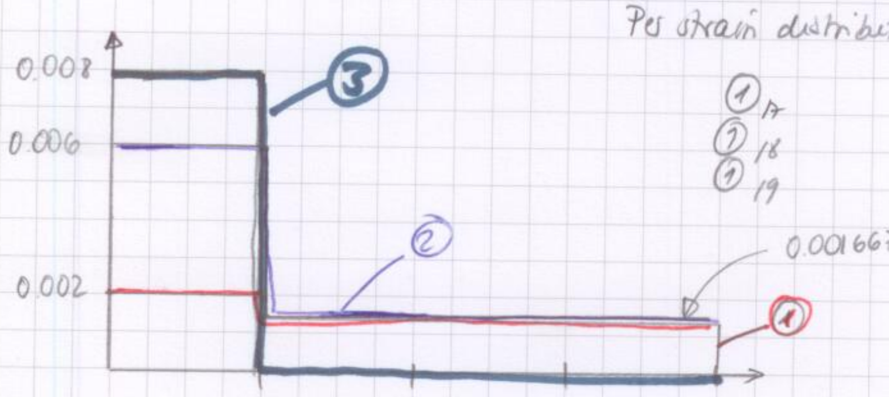
b) Strain distribution

Per strain distribution 1 Pkt



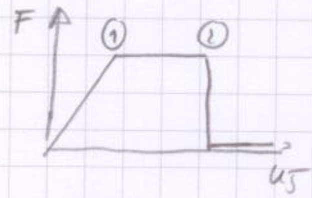
1 17
1 18
1 19

c) Force incrementation method: up to point 1 < afterwards the force does not increase anymore

1 20

d) a) Control node 5:

The curve obtained by controlling node 5 would look as follows:



This means the "snap-back" cannot be captured.

5

If written: up to point 2 : 2

b) Control node 2:

With control node 2 the entire & exact curve could be obtained as the slope of node 2 always increases

2 3

Q3: a) 16
b) 3
c) 1
d) 10

Solution Q4

a) $\vec{F} = \begin{Bmatrix} A_x \\ A_y \\ 0 \\ 0 \\ -B/\sqrt{2} \\ +B/\sqrt{2} \\ P \\ 0 \end{Bmatrix}$ $②_{1,2}$ for every incorrect component -0.5

$\vec{F} = \underline{\underline{K}} \cdot \vec{u}$ $①_3$

b) $K_{global} = \begin{bmatrix} 1.35 & 0.35 & 0 & 0 & -1 & 0 \\ & 1.35 & 0 & -1 & 0 & 0 \\ & & 1.5 & 0.5 & -0.5 & -0.5 \\ & & & 1.5 & -0.5 & -0.5 \\ & sym & & & 1.5 & 0.5 \\ & & & & & 0.5 \end{bmatrix} \cdot 10$ $①_4$ $u_{cond} = \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$

c) Inclined support $\underline{u_3 = v_3}$ or $u_3 - v_3 = 0$ $②_{5,6}$

$K_{penalty} = \begin{bmatrix} 1.35 & 0.35 & 0 & 0 & -1 & 0 \\ & 1.35 & 0 & -1 & 0 & 0 \\ & & 1.5+\omega & +0.5-\omega & -0.5 & -0.5 \\ & & 0.5-\omega & 1.5+\omega & -0.5 & -0.5 \\ & & & & 1.5 & 0.5 \\ & & & & & 0.5 \end{bmatrix}$ correct $④_{7-10}$ every mistake -1

$F_p = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \\ 0 \end{Bmatrix}$ $②_{11,12}$ — support modelled with penalty method → does not show up in force vector $①_{13}$

$F_p = K_p \, u$ $\qquad u = inv(K_p) \cdot F_p$

$②_{14,15}$

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d) Lagrange

a)

$$K_L = \begin{bmatrix} 1.35 & 0.35 & 0 & 0 & -1 & 0 & 0 \\ & 1.35 & 0 & -1 & 0 & 0 & 0 \\ & & 1.5 & 0.5 & -0.5 & -0.5 & 1 \\ & sym & & 1.5 & -0.5 & -0.5 & -1 \\ & & & & 1.5 & 0.5 & 0 \\ & & & & & 0.5 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

4 16-20

b) $u_L = \begin{Bmatrix} u_2 \\ v_2 \\ \vdots \\ u_3 \\ v_4 \\ \lambda \end{Bmatrix}$ 2 21, 22

$F = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \\ 0 \end{Bmatrix}$ 1 23

$u_L = inv(K_L) \cdot F$ 2 24, 25

e) u_3 $-\frac{B}{\sqrt{2}} = [-1 \; 0 \; 0 \; 0 \; 1.5 \; 0.5 \; -0.5 \; -0.5] \cdot 10 \begin{Bmatrix} 0 \\ 0 \\ 9.66 \\ -4 \\ -2 \\ -2 \\ 11.66 \\ -15.66 \end{Bmatrix}$

$= -19.7$ kN (should be 20)

$B = \sqrt{2} \cdot 19.7 = 27.86$ kN (support in compression)

$\ulcorner v_3: \frac{B}{\sqrt{2}} = +20.3$ kN $\lrcorner$

1 correct equation 2 26, 27

correct result (value) for B 2 28, 29

sign correct / clear that support in comp 1 30

Q4 a) 3 c) 11 e) 5

b) 1 d) 10