Nonlinear analysis of structures (CIVIL-449)

Final exam

22.1.2020

CLOSED BOOK EXAM

Duration: 3.0h

Name:	______________________________
Sciper No:	______________________________
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Instructions:

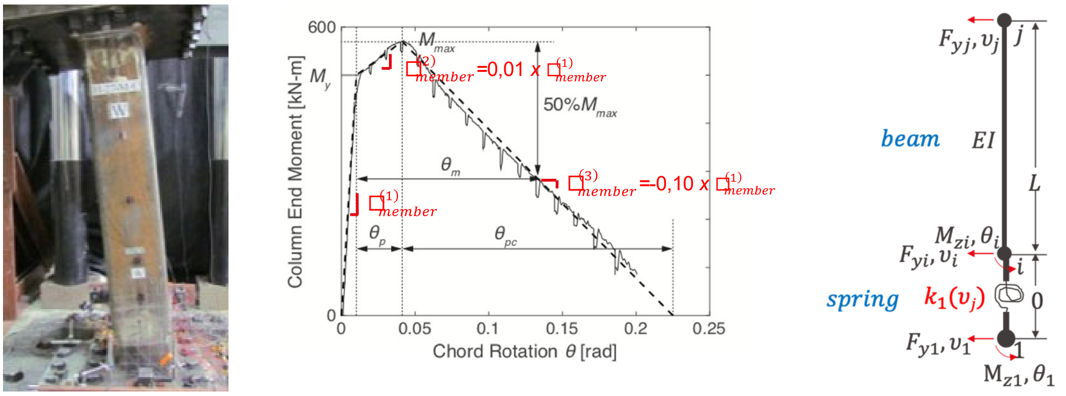
- There are four questions. The total number of points are 130. Question 2 has 10 additional bonus points.
- **Give brief answers** (Short sentences with key words are perfect; no novels please).
- Permitted: 2 pages (= 1 sheet written on both sides), hand-written (not copied or printed)
- Calculator permitted (not the mobile phone; mobile phone must be switched off).

Question 1 (20 points)

Figure 1a shows a cantilever member with length, L, moment of inertia, I, and a Young's modulus, E. Its lateral response is shown in Figure 1b. We would like to approximate the cantilever member with a zero-length rotational element and an elastic beam-column element as shown in Figure 1c. The zero-length element is assumed to be infinitely stiff compared to the elastic beam-column element ("n" times stiffer than the corresponding beam).

Answer the following questions, and support your answers by showing all your derivations:

1. Express the elastic rotational stiffness of the spring, $k_1^{(1)}$ as a function of k, E, I and L.
2. Express the post-yield hardening ratio, a_s, of the spring as a function of n.
3. Express the post-capping hardening ratio, a_c, of the spring as a function of n.



(a) Cantilever member (b) lateral response (c) model idealization

Figure 1: Cantilever member and model idealization

Question 2 (50 points + 10 bonus points)

Part A

1. Provide at least three shortcomings of displacement-based beam-column elements.
2. Explain why the state determination of force-based beam-column elements is challenging compared to displacement-based elements?

Part B

The tapered beam shown in Figure 2 has a linear elastic material. The material modulus of elasticity E is constant. The beam cross-section is rectangular, and the depth changes linearly from $2d$ at the fixed support to d at the tip. The beam width, b, is constant. This beam is analyzed with a single displacement-based beam-column element with two nodes. The left note is *Node-i* and the right node is *Node-j* as shown in the figure.

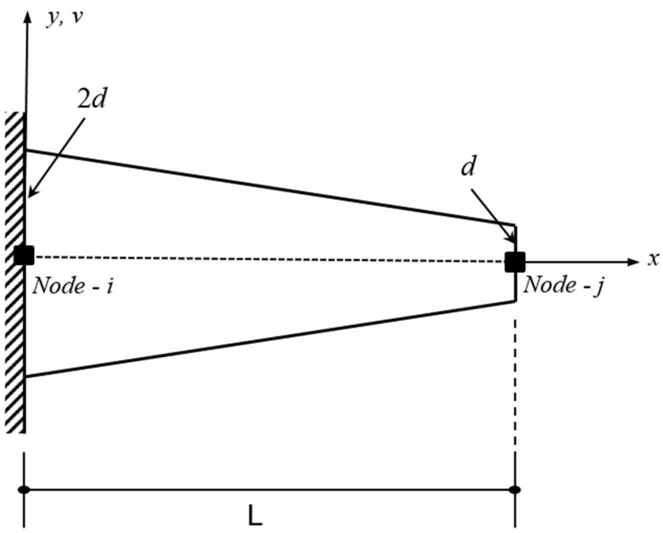


Figure 2: tapered element

The transverse displacement field $v(x)$ along the beam is approximated by the Euler-Bernoulli beam theory assumptions and,

$$v(x) = \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] v_j + \left[\frac{x^3}{L^2} - \frac{x^2}{L}\right] \theta_j \tag{1}$$

From the above displacement field, the curvature field $k(x)$ can be calculated to get the following transformation matrix,

$$k(x) = \mathbf{B(x)} \cdot \begin{Bmatrix} v_j \\ \theta_j \end{Bmatrix} \tag{2}$$

With the use of the principle of virtual displacement method, the resulting stiffness matrix of the element is 2x2 and can be calculated as:

$$\mathbf{k} = \int_0^L [\mathbf{B(x)}]^T \mathbf{k}_s(x)\, [\mathbf{B(x)}] dx \tag{3}$$

Where $\mathbf{B}(\mathbf{x})$ is the strain-displacement matrix, and $\mathbf{k}_s(x) = EI(x)$ is the section stiffness matrix.

Answer the following questions:

1. What type of numerical integration method do you propose in order to calculate the above stiffness matrix "numerically exact"? How many integration points should be used with this method and explain why?
2. Calculate the "numerically exact" stiffness matrix. Depending on the choice of numerical integration you can use the respective tables from the Annex.
3. Is this stiffness matrix the "theoretically exact" stiffness matrix for this tapered beam? Explain why or why not?
4. **Bonus:** Calculate the "theoretically exact" stiffness matrix (**10 extra points**)

Question 3: Bar elements subjected to a tension force (30 points)

A tensile test on a steel bar is modelled here with 4 bar elements, each of 0.1 m length. The bar contains a small defect in element 1. The constitutive relationship of bar element 1 is therefore different to the constitutive relationship of bar element 2, 3 and 4. The constitutive relationship of bar element 1 is $\sigma_2(\varepsilon)$ and that of the other elements $\sigma_1(\varepsilon)$. The relationships are plotted in Figure 4 and the values tabulated in Table 1.

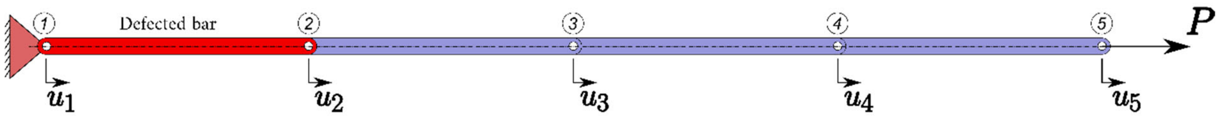


Figure 3: Finite element model with 4 bars for simulating a tensile test on a bar with a small defect.

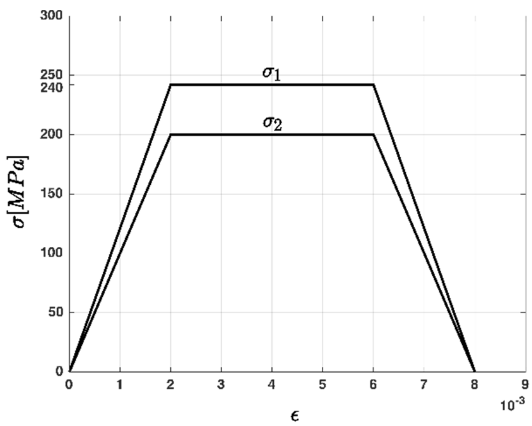


Figure 4 : Constitutive relationship of the four bar elements. Bar element 1 has a yield strength of 200 MPa and bar elements 2-4 of 240 MPa.

Table 1 : Constitutive relationship of the four bar elements.

Strain	Relationship $\sigma_2(\varepsilon)$	Relationship $\sigma_1(\varepsilon)$
	[MPa]	[MPa]
0	0	0
0.002	200	240
0.006	200	240
0.008	0	0

a) Compute and sketch the force-displacement relationship *P* vs u_5 where u_5 is the horizontal displacement of Node 5 and *P* is the horizontal force applied in Node 5. Provide all characteristic values of the curve (i.e., the force and displacement for each point where the slope of the curve changes).

b) Sketch for each segment of the force-displacement relationship one strain distribution along the bar. Pick for this purpose an arbitrary point along the segment and plot the strain distribution for this point.

c) If a force-controlled method is used, up to which point could the force-displacement relationship be computed? Explain why.

d) If a displacement-controlled method is used, up to which point could the force-displacement relationship be computed if

   a. The control node was Node 5? Explain why.

   b. The control node was Node 2? Explain why.

Question 4: Truss with inclined support (30 points)

The truss shown in Figure 5 consists of 4 nodes and 5 bars. It is subjected to a horizontal force that is applied in Node 4. Node 1 is pinned. Node 3 has an inclined sliding support. The global stiffness matrix of the truss has been assembled and reads as follows:

$$K_{global} = \begin{bmatrix} 1.35 & 0.35 & -0.35 & -0.35 & -1.00 & 0 & 0 & 0 \\ 0.35 & 0.35 & -0.35 & -0.35 & 0 & 0 & 0 & 0 \\ -0.35 & -0.35 & 1.35 & 0.35 & 0 & 0 & -1.00 & 0 \\ -0.35 & -0.35 & 0.35 & 1.35 & 0 & -1.00 & 0 & 0 \\ -1.00 & 0 & 0 & 0 & 1.50 & 0.50 & -0.50 & -0.50 \\ 0 & 0 & 0 & -1.00 & 0.50 & 1.50 & -0.50 & -0.50 \\ 0 & 0 & -1.00 & 0 & -0.50 & -0.50 & 1.50 & 0.50 \\ 0 & 0 & 0 & 0 & -0.50 & -0.50 & 0.50 & 0.50 \end{bmatrix} \cdot 10^1 \text{ [kN/mm]}$$

The displacement vector u has the following components:

$$u = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}$$

where u_i and v_i are the displacement in x-direction and y-direction of Node i, respectively. The horizontal force P applied in Node 4 is 20 kN.

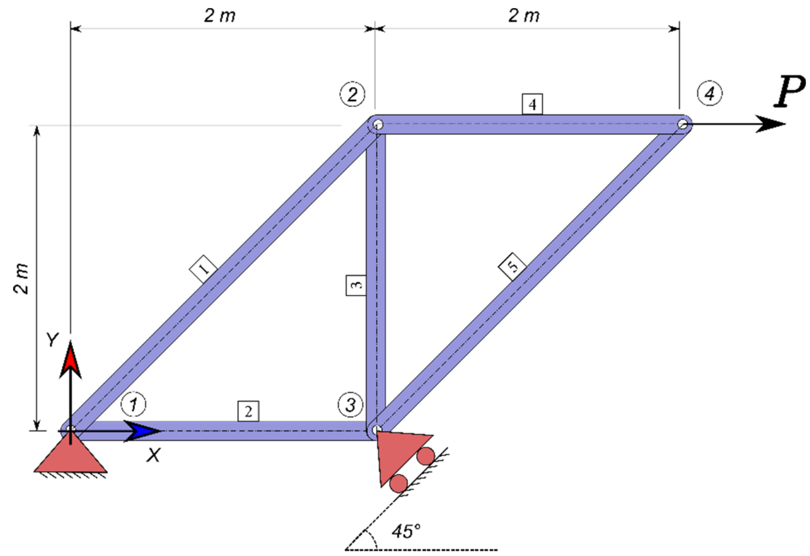


Figure 5: Truss with inclined support in Node 3

a) Write the force vector and the global equilibrium equation.
b) Derive the condensed stiffness matrix accounting for the fact that Node 1 is pinned. Continue working with this stiffness matrix.

c) Model the inclined support using the Penalty method:
   a. Write down the modified stiffness matrix.
   b. Write down the equation that would need to be solved to determine the displacements.

d) Model the inclined support using the Lagrange Multiplier method:
   a. Write down the modified stiffness matrix.
   b. Write down the equation that would need to be solved to determine the displacements.

e) The resulting displacement vector is the following:

$$u = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 9.66 \\ -4.00 \\ +2.00 \\ -2.00 \\ 11.66 \\ -15.66 \end{pmatrix} mm$$

Compute the reaction force at the inclined support (Node 3).

ANNEX

Gauss quadrature

Gauss Points *n*	*Point* r_i	*Weight Coefficient* w_i	Polynomial Order *m*
1	0	2	1
2	-1/√3, 1/√3	1, 1	3
3	-√0.6, 0, √0.6	5/9, 8/9, 5/9	5
4	-0.861136, -0.339981, 0.339981, 0.861136	0.347855, 0.652145, 0.652145, 0.347855	7
5	-0.906180, -0.538469, 0, 0.538469, 0.906180	0.236927, 0.478629, 0.568889, 0.478629, 0.236927	9
6	-0.932470, -0.661209, -0.238619, 0.238619, 0.661209, 0.932470	0.171324, 0.360762, 0.467914, 0.467914, 0.360762, 0.171324	11

Gauss Lobatto & Gauss Radu

Gauss Lobatto

n	x_i	x_i	w_i	w_i
3	0	0.00000	$\frac{4}{3}$	1.333333
	± 1	$\pm$1.00000	$\frac{1}{3}$	0.333333
4	$\pm\frac{1}{5}\sqrt{5}$	$\pm$0.447214	$\frac{5}{6}$	0.833333
	± 1	$\pm$1.000000	$\frac{1}{6}$	0.166667
5	0	0.000000	$\frac{32}{45}$	0.711111
	$\pm\frac{1}{7}\sqrt{21}$	$\pm$0.654654	$\frac{49}{90}$	0.544444
	± 1	$\pm$1.000000	$\frac{1}{10}$	0.100000
6	$\sqrt{\frac{1}{21}\left(7-2\sqrt{7}\right)}$	$\pm$0.285232	$\frac{1}{30}\left(14+\sqrt{7}\right)$	0.554858
	$\sqrt{\frac{1}{21}\left(7+2\sqrt{7}\right)}$	$\pm$0.765055	$\frac{1}{30}\left(14-\sqrt{7}\right)$	0.378475
	± 1	$\pm$1.000000	$\frac{1}{15}$	0.066667

Gauss Radu

n	x_i	w_i
2	-1	0.5
	0.333333	1.5
3	-1	0.222222
	-0.289898	1.02497
	0.689898	0.752806
4	-1	0.125
	-0.575319	0.657689
	0.181066	0.776387
	0.822824	0.440924
5	-1	0.08
	-0.72048	0.446208
	-0.167181	0.623653
	0.446314	0.562712
	0.885792	0.287427