School of Architecture, Civil and Environmental Engineering
CIVIL ENGINEERING INSTITUTE
EPFL – RESILIENT STEEL STRUCTURES LABORATORY

Course: CIVIL 449 - Nonlinear analysis of structures

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Assignment #3: Nonlinear analysis of structures considering material and geometric nonlinearities

Q1 (65 points): Extend the program you wrote for Assignment #2 to conduct nonlinear analysis of frame structures by considering both geometric and material nonlinearities. The program can be written at any programming language of your preference. Your program should consider the following:

- Zero-length rotational spring with elastic beam-column elements
- Displacement-based beam-column elements

Your program should be able to determine the nodal displacements, member forces and support reactions for planar frames by nonlinear analysis. Assume that the members are all prismatic, i.e., the axial and flexural rigidities of the members are constant along their length.

NOTE 1: The constitutive formulation to be considered in the rotational spring (moment – rotation relation) should be that of in-class exercise #8 (Week #8 → Material nonlinearity and concentrated plasticity)

NOTE 2: The constitutive formulation to be considered in the fibers of your cross section may be the same with that developed for the in-class exercise #8, without the softening path (you can assume a very large θ_{pc}). In this case, you should be using this as a stress-strain relation.

Q2 (35 points): Use your program from Q1 to compute the following:

For $P = 0.5P_{cr}$ compare the total base shear versus lateral displacement equilibrium paths for the following cases:

a. Case 1: Elastic material and linear geometry analysis
b. Case 2: Nonlinear geometric analysis (elastic material)
c. Case 3: Nonlinear material analysis (inelastic material but elastic geometry)
    i. Case 3a by using a zero-length element
    ii. Case 3b by using a displacement-based element
d. Case 4: Nonlinear analysis for both material and geometry
    i. Case 4a by using a zero-length element
    ii. Case 4b by using a displacement-based element

Assume the following geometry for the members:
-Beams: rectangular cross section with width $b_b = 300mm$ and a height $h_b = 700mm$ (strong axis bending)
-Columns: square section with width $b_c = 300mm$ and a height $h_c = 300mm$

Comment on your results based on the choice of the analysis and iterative method(s) (e.g., displacement / load control).
Assume that the cross sections are made of S355 steel ($f_y = 355MPa$, $E = 200GPa$)

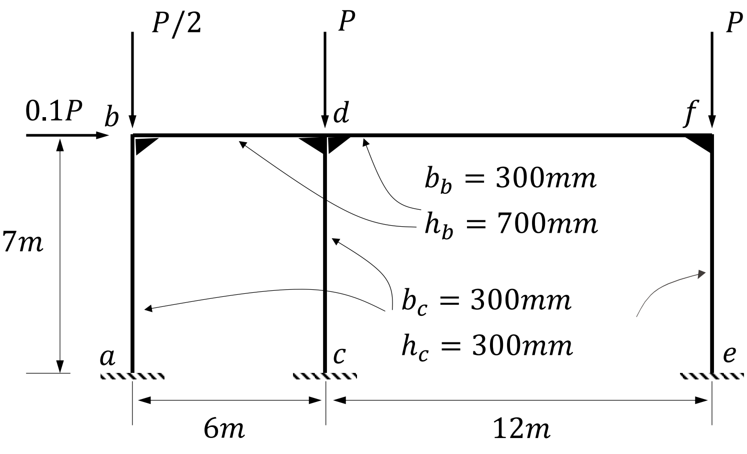

Figure 1. Planar frame

NOTE 1: For case 3a and 4a assume the following model parameters:

- $\theta_p = \theta_c - \theta_y = 0{,}02\, rad$
- $\theta_{pc} = \theta_u - \theta_c = 0{,}05\, rad$
- $M_y^* = W_{el,y} \cdot f_y$
- $M_u = 1{,}1M_y^*$

NOTE 2: For case 3b and 4b assume 3% strain hardening ratio in the assumed stress-strain relationship

Solution:

Elastic beam-column element with two inelastic rotational springs at the member ends:
Displacement-control is used because the inelastic spring exhibits a snap-through behavior.

To determine the load-displacement relation of the moment frame when it is modeled using an elastic beam-column element with two zero-length inelastic flexural springs, the same steps are used as shown in the solution of Exercise 3 of Week 8.

The direct method is used to enforce that the first translational degree of freedom of the two nodes of the spring element is equal, and similarly for the second translational degree of freedom.

The following global degrees of freedom are used

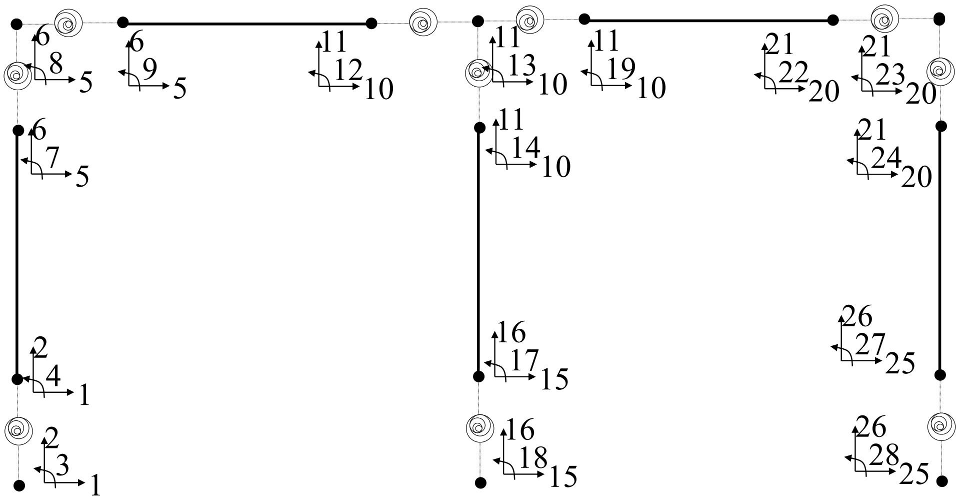

Figure 2. Global degrees of freedom with equal DOF constraint – elastic beam-column element with inelastic rotational springs modeling approach

The spring rotation θ is defined as $\theta = v_J - v_I$, where I and J denote the inner and outer nodes of the spring (i.e., the inner node refers to the one connected to the elastic beam-column element). This convention is very important when the constitutive relation assigned to the spring exhibits an asymmetric response in tension and compression. This rotation is then the input for the spring constitutive formulation, which returns the spring moment M_{spring} and tangent stiffness k_{spring}. The former is used to form the spring resisting force vector $\boldsymbol{F}_{int,spring}$ and the latter to form the stiffness matrix $\boldsymbol{K}_{spring}$ as follows:

$$\boldsymbol{F}_{int,spring} = \begin{pmatrix} -M_{spring} \\ M_{spring} \end{pmatrix} \tag{1}$$

$$\boldsymbol{K}_{spring} = k_{spring} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{2}$$

These quantities are then used when assembling the structure resisting force vector and stiffness matrix.

For the elastic-beam column element, the following tangent stiffness matrix is derived using a procedure similar to that discussed in Question 1 of Week 8:

$$\hat{\mathbf{k}}_{mod} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{S_{22}EI_e}{L} & \frac{S_{23}EI_e}{L} \\ 0 & \frac{S_{32}EI_e}{L} & \frac{S_{33}EI_e}{L} \end{bmatrix} \tag{3}$$

With

$$S_{22} = S_{33} = \frac{12n+6}{3n+2} \tag{4}$$

$$S_{23} = S_{32} = \frac{6n+1}{3n+2} \tag{5}$$

Where n is the stiffness modification factor

To determine the secondary equilibrium path for $\alpha = 0.1$ with this modeling approach, the following steps are used:

1) Define the member properties (both for the springs and elastic beam-column elements)

2) Define the connectivity and mapping matrices between local and global degrees of freedom. The direct method is used to enforce the equal DOF constraints

3) For each member, determine the transformation matrix $\boldsymbol{T}$ between local and global coordinates. In this exercise, the local x-axis is defined in the axial direction of the element; therefore, it corresponds to the global Y-axis

4) Assemble the initial structure stiffness matrix $\boldsymbol{K}_{structure}$

5) Define the boundary conditions, the external loads (i.e., apply the reference load $\boldsymbol{F}^{ref}$), the fixed and the free degrees of freedom of the problem

6) Initialize the variables used within the Newton-Raphson procedure

$$\lambda = 0, \boldsymbol{v} = \boldsymbol{0}$$

Where λ denotes the load multiplier (i.e. $\boldsymbol{F}_{ext} = \lambda \boldsymbol{F}^{ref}$)

7) Define the parameters defining the displacement-control algorithm:
- The DOF at which the displacement-control algorithm is imposed
- The number of steps n_{tot}
- The imposed displacement at every step $\Delta\bar{u}$
- The tolerance tol
- The maximum number of iterations per iteration of the Newton-Raphson loop $i_{\max}$

8) For load increment n, perform the Newton-Raphson iterations

8.1) For $i = 1$, set, $\Delta\boldsymbol{F}_{ext}^{n,i=1} = \Delta\bar{\lambda}\boldsymbol{F}^{ref}$, $\boldsymbol{F}_{int}^{n,i=1} = \boldsymbol{F}_{int}^{n-1}$, $\boldsymbol{K}_{structure}^{n,1} = \boldsymbol{K}_{structure}^{n-1}$ and $\boldsymbol{v}^{n,1} = \boldsymbol{v}^{n-1}$

8.2) Compute the increment in structure displacements $\Delta\boldsymbol{v}^{n,i}$:

$$\Delta\boldsymbol{v}_f^{n,i} = \left(\boldsymbol{K}_{structure,f}^{n,i}\right)^{-1} \Delta\boldsymbol{F}_{ext}^{n}$$

Where the subscript f denotes the free degrees of freedom of the system

8.3) Update the structure displacements:

$$\boldsymbol{v}^{n,i} = \boldsymbol{v}^{n,i-1} + \Delta\boldsymbol{v}^{n,i}$$

8.4) Assemble the structure material and geometric stiffness matrices $\boldsymbol{K}_{e,structure}^{n,i}$ and $\boldsymbol{K}_{g,structure}^{n,i}$, as well as the structure resisting force vector $\boldsymbol{F}_{int}^{n,i}$. With a loop, go over all elements (springs and elastic beam-column elements):

8.4.1) Determine the element displacement vector in the local reference frame $\boldsymbol{u}^{n,i}$

$$\boldsymbol{u}_{elem}^{n,i} = \boldsymbol{T}_{elem}\boldsymbol{v}_{elem}^{n,i}$$

Where the subscript $elem$ denotes the DOFs corresponding to element $elem$.

8.4.2) For the elastic beam-column element: using the corotational formulation or the linear formulation, compute the element displacements in the basic reference frame $\overline{\boldsymbol{u}} = [\bar{u}_1, \bar{u}_2, \bar{u}_3]^T$.

For the inelastic rotational springs, the rotations in the local reference frame are directly used in the constitutive relation

8.4.3) Compute the element internal forces in the basic reference frame $\overline{\boldsymbol{q}}^{n,i}$:
For the elastic beam-column element:

$$\overline{\boldsymbol{q}}^{n,i} = \overline{\boldsymbol{K}}^{n,i}\overline{\boldsymbol{u}}^{n,i}$$

For the inelastic rotational springs: use the constitutive relation

8.4.4) Compute the element internal force vector in the local reference frame:
For the elastic beam-column element:

$$\boldsymbol{Q}_{elem}^{n,i} = \left(\boldsymbol{L}^{n,i}\right)^T\overline{\boldsymbol{q}}^{n,i}$$

Where $\boldsymbol{L}^{n-1}$ is the transformation matrix from the basic to the local reference frame.

For the inelastic rotational springs: the resisting moment obtained from the constitutive relation is directly expressed in the local reference frame

8.4.5) For the elastic beam-column element, determine the element geometric stiffness matrix in the local reference frame $\boldsymbol{K}_{g,elem}^{n,i}$.
For the inelastic rotational springs, there is no element geometric stiffness matrix

8.4.6) Assemble the structure material and geometric stiffness matrices $\boldsymbol{K}_{e,structure}^{n,i}$ and $\boldsymbol{K}_{g,structure}^{n,i}$ as well as the structure internal force vector $\boldsymbol{F}_{int}^{n,i}$ with the element quantities

8.5) Compute the unbalanced load vector $\boldsymbol{F}_{unb}^{n,i} = \boldsymbol{F}_{int}^{n,i} - \boldsymbol{F}_{ext}^{n}$

8.6) Check if the Newton-Raphson procedure has converged. In the code provided with the solution, convergence is achieved once

$$\frac{\left\|\boldsymbol{F}_{unb,f}^{n,i}\right\|}{\left\|\boldsymbol{F}_{ext,f}^{n,i}\right\|} < tol$$

8.7) If iteration i has converged, go to next load step n, else set $i = i + 1$ and $\Delta\boldsymbol{F}_{ext}^{n,i} = -\boldsymbol{F}_{unb,f}^{n,i-1}$ and go to the next step

Displacement-based beam-column element:
Each beam and column member of the moment frame is modeled using several displacement-based beam-column elements. This is done because of the assumed shape functions of the displacement-based element. If we were to use a force-based element, a single element would suffice to idealize a member.

For each one of the displacement-based beam-column elements, five integration sections are placed along the element length following the Gauss-Lobatto integration rule. At each of these sections, the cross section is discretized using a certain number of fibers. Each fiber is assigned with the bilinear uniaxial constitutive formulation representing the engineering stress-strain at the material scale.

The following figure shows the elements used in this modeling approach:

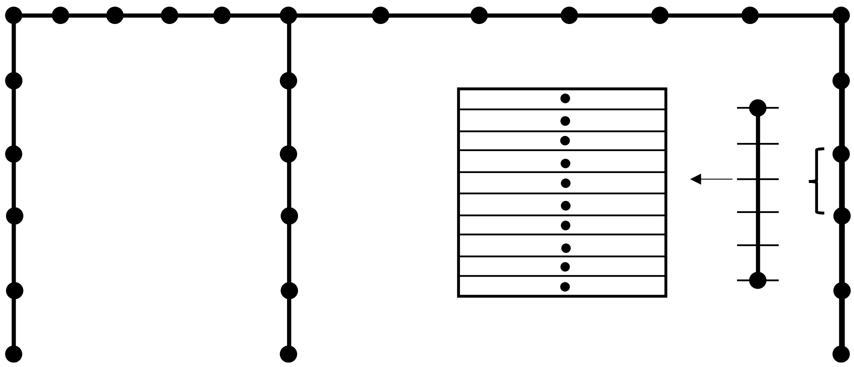

Figure 3. Finite element model using displacement-based beam-column elements

To determine the secondary equilibrium path for $\alpha = 0.1$ with this modeling approach, the following steps are used:

1) Define the member properties:

- Number of displacement-based beam-column elements for each member
- Numerical integration rule along the element length: location and weight of each integration points
- Fiber section

For the solution presented here, each beam and column are modeled using 5 displacement-based beam-column elements. Five integration points are placed along the length for each element, and the section is discretized using 10 fibers

2) Define the connectivity matrix and the mapping matrix between local and global degrees of freedom

3) For each member, determine the transformation matrix $\boldsymbol{T}$ between local and global coordinates. In this assignment, the local x-axis is defined in the axial direction of the element; therefore, it corresponds to the global Y-axis

4) Assemble the initial stiffness matrix $\boldsymbol{K}_{structure}$ of the structure

5) Define the boundary conditions, the external loads (i.e., apply the reference load $\boldsymbol{F}^{ref}$), the fixed and the free degrees of freedom of the problem

6) Initialize the variables used within the Newton-Raphson procedure

$$\lambda = 0, \boldsymbol{v} = \mathbf{0}$$

Where λ denotes the load multiplier (i.e. $\boldsymbol{F}_{ext} = \lambda \boldsymbol{F}^{ref}$)

7) Define the parameters defining the displacement-control algorithm:

- The DOF at which the displacement-control algorithm is imposed
- The number of steps n_{tot}
- The imposed displacement at every step $\Delta\bar{u}$
- The tolerance tol
- The maximum number of iterations for each iterations of the Newton-Raphson loop $i_{\max}$

8) For load increment n, perform the Newton-Raphson iterations

8.1) For $i = 1$, set, $\Delta\boldsymbol{F}_{ext}^{n,i=1} = \Delta\bar{\lambda}\boldsymbol{F}^{ref}$, $\boldsymbol{F}_{int}^{n,i=1} = \boldsymbol{F}_{int}^{n-1}$, $\boldsymbol{K}_{structure}^{n,1} = \boldsymbol{K}_{structure}^{n-1}$ and $\boldsymbol{v}^{n,1} = \boldsymbol{v}^{n-1}$

8.2) Compute the increment in structure displacements $\Delta\boldsymbol{v}^{n,i}$:

$$\Delta\boldsymbol{v}_f^{n,i} = \left(\boldsymbol{K}_{structure,f}^{n,i}\right)^{-1} \Delta\boldsymbol{F}_{ext}^{n}$$

Where the subscript f denotes the free degrees of freedom of the system

8.3) Update the structure displacements:

$$\boldsymbol{v}^{n,i} = \boldsymbol{v}^{n,i-1} + \Delta\boldsymbol{v}^{n,i}$$

8.4) Assemble the structure material and geometric stiffness matrices $\boldsymbol{K}_{e,structure}^{n,i}$ and $\boldsymbol{K}_{g,structure}^{n,i}$, as well as the structure resisting force vector $\boldsymbol{F}_{int}^{n,i}$. With a loop, go over all elements:

8.4.1) Determine the element displacement vector in the local reference frame $\boldsymbol{u}^{n,i}$

$$\boldsymbol{u}_{elem}^{n,i} = \boldsymbol{T}_{elem}\boldsymbol{v}_{elem}^{n,i}$$

Where the subscript $elem$ denotes the degrees of freedom corresponding to element $elem$.

8.4.2) Using the corotational formulation or the linear formulation, compute the element displacements in the basic reference frame $\bar{\boldsymbol{u}} = [\bar{u}_1, \bar{u}_2, \bar{u}_3]^T$.

8.4.3) Perform the element state determination procedure, i.e., compute the element internal forces in the basic reference frame $\bar{\boldsymbol{q}}^{n,i}$ and the element tangent stiffness matrix in the basic reference frame $\bar{\boldsymbol{K}}^{n,i}$:

8.4.3.1) For every section along the element length: compute the section displacement vector $\mathbf{d}_s^{n,i}$

$$\mathbf{d}_s^{n,i} = \bar{\mathbf{B}}(x)\bar{\mathbf{u}}^{n,i}$$

Where $\bar{\mathbf{B}}(x)$ is the matrix with the displacement interpolation functions

8.4.3.2) Perform the section state determination procedure, i.e., compute the section tangent stiffness matrix $\mathbf{k}_s^{n,i}$ and the section resisting force vector: $\mathbf{Q}_{sr}^{n,i}$:

Iterate over each fiber $iFib$ of the section:

8.4.3.2.1) Compute the fiber strain:

$$\varepsilon_{ifib} = \mathbf{l}_{ifib}\mathbf{d}_s$$

8.4.3.2.2) Perform the material state determination: input the fiber strain ε_{ifib} into the constitutive law and return the fiber stress σ_{ifib} and tangent modulus k_{ifib}

8.4.3.3) Integrate the stress and tangent moduli of all the fibers to form the section resisting force vector $\mathbf{Q}_{sr}^{n,i}$ and tangent stiffness matrix $\mathbf{k}_s^{n,i}$:

$$\mathbf{Q}_{sr}^{n,i} = \sum_{k=1}^{nfib} \mathbf{l}_{ifib}^T \cdot (\sigma_{ifib}A_{ifib})$$

$$\mathbf{k}_s^{n,i} = \sum_{k=1}^{nfib} \boldsymbol{l}_{ifib}^T \cdot (k_{ifib}A_{ifib}) \cdot \boldsymbol{l}_{ifib}$$

8.4.4) Integrate the section resisting force vector $\mathbf{Q}_{sr}^{n,i}$ and tangent stiffness matrix $\mathbf{k}_s^{n,i}$ to compute the element internal forces in the basic reference frame $\overline{\boldsymbol{q}}^{n,i}$ and the element tangent stiffness matrix in the basic reference frame $\overline{\boldsymbol{K}}^{n,i}$

$$\overline{\boldsymbol{q}}^{n,i} = \int_0^L \overline{\boldsymbol{B}}^T(x) \cdot \boldsymbol{Q}_{sr}^{n,i}(x) \cdot dx \approx \sum_{iSec=1}^{nSec} \overline{\mathbf{B}}^T(x_{iSec}) \cdot \boldsymbol{Q}_{sr}^{n,i}(x_{iSec}) \cdot w_{iSec}$$

$$\overline{\boldsymbol{K}}^{n,i} = \int_0^L \overline{\mathbf{B}}^T(x) \cdot \mathbf{k}_s^{n,i} \cdot \overline{\mathbf{B}}(x) \cdot dx$$

$$\approx \sum_{iSec=1}^{nSec} \overline{\mathbf{B}}^T(x_{iSec}) \cdot \mathbf{k}_s^{n,i}(x_{iSec}) \cdot \overline{\mathbf{B}}(x_{iSec}) \cdot w_{iSec}$$

Where x_{iSec} and w_{iSec} denote the location and weight of the quadrature point $iSec$, respectively

8.4.5) Compute the element internal force vector in the local reference frame:

$$\boldsymbol{Q}_{elem}^{n,i} = \left(\boldsymbol{L}^{n,i}\right)^T \overline{\boldsymbol{q}}^{n,i}$$

Where $\boldsymbol{L}^{n-1}$ is the transformation matrix from the basic to the local reference frame

8.4.6) Determine the element geometric stiffness matrix in the local reference frame $\boldsymbol{K}_{g,elem}^{n,i}$

8.4.7) Assemble the structure material and geometric stiffness matrices $\boldsymbol{K}_{e,structure}^{n,i}$ and $\boldsymbol{K}_{g,structure}^{n,i}$ as well as the structure internal force vector $\boldsymbol{F}_{int}^{n,i}$ with the element quantities

8.5) Compute the unbalanced load vector $\boldsymbol{F}_{unb}^{n,i} = \boldsymbol{F}_{int}^{n,i} - \boldsymbol{F}_{ext}^{n}$

8.6) Check if the Newton-Raphson procedure has converged. In the code provided with the solution, convergence is achieved once

$$\frac{\left\| \boldsymbol{F}_{unb,f}^{n,i} \right\|}{\left\| \boldsymbol{F}_{ext,f}^{n,i} \right\|} < tol$$

8.7) If iteration i has converged, go to next load step n, else set $i = i + 1$ and $\Delta \boldsymbol{F}_{ext}^{n,i} = -\boldsymbol{F}_{unb,f}^{n,i-1}$ and go to step

Comparison of various modeling approaches:

The following figure compares the results obtained for various analysis methods:

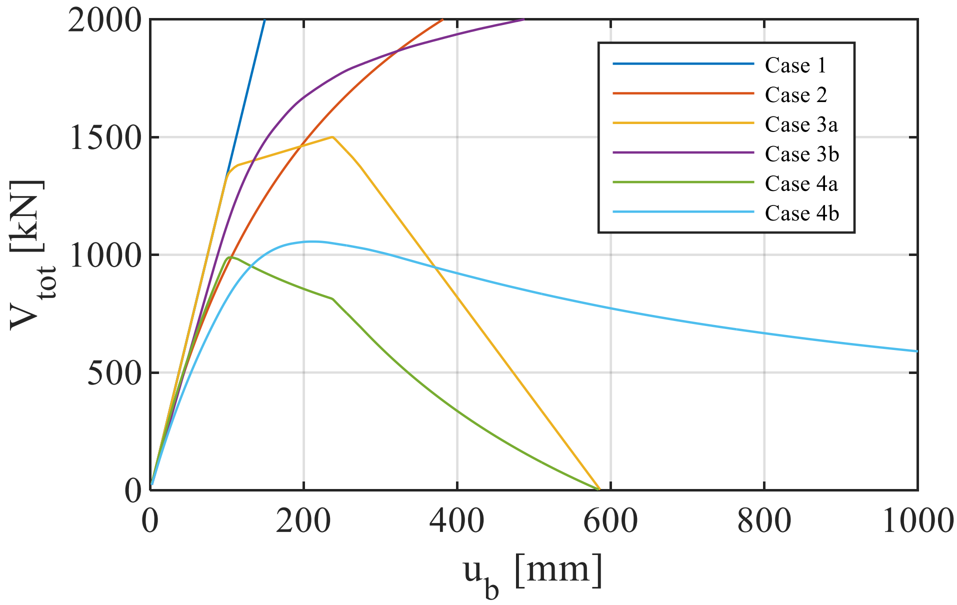

Figure 4. Comparison of computed secondary equilibrium paths under various analysis methods

Case 1: Elastic material and linear geometry analysis
Case 2: Nonlinear geometric analysis (elastic material)
Case 3: Nonlinear material analysis (inelastic material but elastic geometry)
  Case 3a by using a zero-length element
  Case 3b by using a displacement-based element
Case 4: Nonlinear analysis for both material and geometry
  Case 4a by using a zero-length element
  Case 4b by using a displacement-based element