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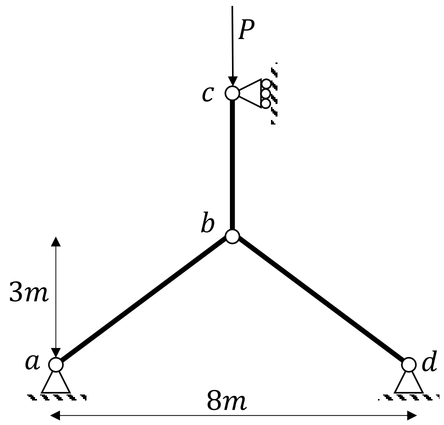
In-class Exercise Week #6: Load-displacement constraint methods

Exercise #1:

Consider the following truss structure:



The members have the followings properties:

- $ab: E = 200\ GPa, l_{o,ab} = 5m$, square cross section of dimension $b_{ab} = 500mm$
- $bc: E = 200\ GPa, l_{o,bc} = l_{o,ab}$, square cross section of dimension b_{bc}
- $bd: E = 200\ GPa, l_{o,bd} = l_{o,ab}$, square cross section of dimension $b_{bd} = b_{ab}$

1) Analytically, determine the minimum dimension bc_{min} such that the structure does not exhibit a snap-back response in the $P - v_c$ equilibrium path (v_c denotes the vertical displacement at node c).

Hint: Consider the tangent in the $P - v_c$ equilibrium path once members ab and bd are horizonal (i.e., when $P = 0$).

2) Derive the corotational formulation for a 2d truss element

3) Model the structure using 2d truss elements and simulate the response of the structure in the case where node c is displaced vertically up to $v_c = 8.0m$ downwards:
a) Using displacement control for $b_{bc} = 2 \cdot b_{ab}$
b) Using arc-length control for $b_{bd} = b_{ab}/2$

Solution:

1) From the system of equations given in Slide 38 of Week 5, the relation between the external load P and the vertical displacement v_b at point b is given by

$$P = \frac{2EA_{ab}}{l_o}\left(\frac{l_o}{\sqrt{l_o^2 - 2v_b l_o sin(\theta_0) + v_b^2}} - 1\right)(\mathrm{l_o}\sin(\theta_0) - v_b) \tag{1}$$

Where $l_o = l_{o,ab} = l_{o,bc} = l_{o,bd}$. With respect to the equations given in Slide 38 of Week 5, the dimensionless displacement a_1 corresponds to v_b/l_o.

As stated in the hint, snap-back in the $P - v_c$ equilibrium path occurs once members ab and bd become horizonal. At this point, the load $P = 0$ and $v_b = H = 3.0m$.

From the system of equations given in Slide 38 of Week 5, the relation between the vertical displacement v_b and v_c is given by

$$P = \frac{EA_{bc}}{l_o}(v_c - v_b) \tag{2}$$

For brevity, let's define $k_{bc} = \frac{EA_{bc}}{l_o}$

Therefore, the vertical displacement at point c is given by

$$v_c = \frac{P}{k_{bc}} + v_b \tag{3}$$

Equation 1 can be rewritten using Equation 3. This gives

$$P = \frac{2EA_{ab}}{l_o}\left(\frac{l_o}{\sqrt{l_o^2 - 2\left(v_c - \frac{P}{k_{bc}}\right) l_o sin(\theta_0) + \left(v_c - \frac{P}{k_{bc}}\right)^2}} - 1\right)\left(\mathrm{l_o}\sin(\theta_0) - v_c + \frac{P}{k_{bc}}\right) \tag{4}$$

Differentiating Equation 4 with respect to P gives

$$1 = \frac{2EA_{ab}}{l_o}\left(\frac{l_o}{\sqrt{l_o^2 - 2\left(v_c - \frac{P}{k_{bc}}\right) l_o sin(\theta_0) + \left(v_c - \frac{P}{k_{bc}}\right)^2}} - 1\right)\left(\frac{1}{k_{bc}} - \frac{\partial v_c}{\partial P}\right) +$$
$$\frac{2EA_{ab}}{l_o}\left(\mathrm{l_o}\sin(\theta_0) - v_c + \frac{P}{k_{bc}}\right)\left(-\frac{l_o}{2}\left[l_o^2 - 2\left(v_c - \frac{P}{k_{bc}}\right) l_o sin(\theta_0) + \left(v_{\square\square} - \frac{P}{k_{bc}}\right)^2\right]^{-\frac{3}{2}}\right)\cdot$$
$$\left(-2\left(\frac{\partial v_c}{\partial P} - \frac{1}{k_{bc}}\right) l_o \sin(\theta_0) + 2\left(\frac{\partial v_c}{\partial P} - \frac{1}{k_{bc}}\right)\left(v_c - \frac{P}{k_{bc}}\right)\right) \tag{5}$$

From Equation 3, it is evident that $P = 0$ corresponds to $v_b = v_c$. Therefore, Snap-back occurs when $P = 0$ and $v_c = v_b = H$.
Moreover, a snap-back response in the $P - v_c$ equilibrium path occurs when $\frac{\partial v_c}{\partial P} = 0$.

Therefore, solving Equation 5 with $\frac{\partial v_c}{\partial P} = 0$, $P = 0$ and $v_c = H$ gives

$$k_{bc,cr} = \frac{2EA_{ab}}{l_o}\left(\frac{1}{\sqrt{1-\sin^2(\theta_0)}} - 1\right) \tag{6}$$

The axial stiffness of member bc is as follows:

$$k_{bc,cr} = \frac{EA_{bc,min}}{l_o} = \frac{Eb_{bc,min}^2}{l_o} = \frac{2Eb_{ab}^2}{l_o}\left(\frac{1}{\sqrt{1-\sin^2(\theta_0)}} - 1\right) \tag{7}$$

Therefore, the minimum dimension bc_{min} such that the structure does not exhibit a snap-back response in the $P - v_c$ equilibrium path is given by

$$b_{bc,min} = \sqrt{2b_{ab}^2\left(\frac{1}{\sqrt{1-\sin^2(\theta_0)}} - 1\right)} = \sqrt{2 \cdot 500^2\left(\frac{1}{\sqrt{1-\left(\frac{3}{5}\right)^2}} - 1\right)} = 354mm \tag{8}$$

For $b_{bc} < b_{bc,min}$, the $P - v_c$ equilibrium path will exhibit a snap-back response, while for $b_{bc} > b_{bc,min}$, the $P - v_c$ equilibrium path will exhibit a snap-trough response.

2) The derivation of the corotational formulation for the 2d beam-column element is presented in Slides 24 to 32 of Week 4. In particular, the rotational degrees of freedom are condensed, i.e., the axial degree of freedom is only considered.

The axial displacement in the basic reference system is given by

$$\bar{u} = L_n - L \tag{9}$$

With

$$L_n = \sqrt{(L + \Delta u_x)^2 + \left(\Delta u_y\right)^2} \tag{10}$$

With

$$\Delta u_y = u_4 - u_2 \tag{11}$$

$$\Delta u_x = u_3 - u_1 \tag{12}$$

Where u denote the element displacements in the local reference system. The following quantities are defined to describe the rigid body rotation:

$$\beta = \arctan\left(\frac{\Delta u_y}{L + \Delta u_x}\right) \tag{13}$$

$$c = \cos(\beta) = \frac{L + \Delta u_x}{L_n} \tag{14}$$

$$s = \sin(\beta) = \frac{\Delta u_y}{L_n} \tag{15}$$

Condensing out the rotational degrees of freedom in the compatibly matrix **L** defined in Slide 27 of Week 4 gives

$$\mathbf{L} = [-c \quad -s \quad c \quad s] \tag{16}$$

Similarly, condensing out the rotational degrees of freedom in the geometric stiffness matrix $\mathbf{K}_{geom}$ defined in Slide 32 of Week 4 gives,

$$\mathbf{K}_{geom} = \frac{\bar{q}_1}{L_n}\begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix} \tag{17}$$

Where $\bar{q}_1$ is the element internal axial force in the basic reference system.

3) Part (a)

To determine the $P - v_c$ equilibrium path using displacement control, the following steps are used:

1) Define the member properties (E, A and l)

2) Define the connectivity matrix and the mapping matrix between local and global degrees of freedom.

The figure below shows the global degrees of freedom used for the structure

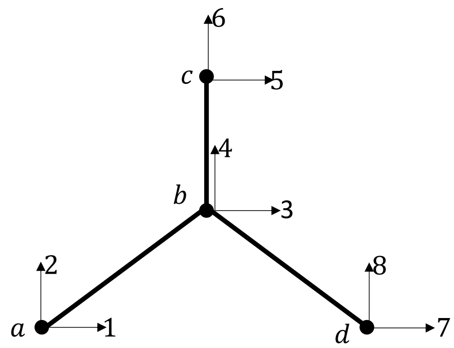


Figure 1. Global degrees of freedom

The mapping matrix $\boldsymbol{numEq}$ is therefore given by

$$\boldsymbol{numEq} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{bmatrix}$$

3) For each member, determine the transformation matrix **T** between local and global coordinates. In this exercise, the local x-axis is defined in the axial direction of the element

4) Assemble the initial structure stiffness matrix $\mathbf{K}_{structure}$

5) Define the boundary conditions, the fixed and the free degrees of freedom of the problem, the external loads (i.e., apply the reference load $\overline{\boldsymbol{F}}_{ext} = [0\ 0\ 0\ 0\ \ 0 -1\ \ 0\ 0]^T$)

6) Initialize the variables used within the displacement-control procedure

$$\lambda = 0, \boldsymbol{v} = \mathbf{0}$$

7) Define the parameters defining the displacement-control algorithm:

- The dof controlling the displacement $q_{dof} = 6$
- The final displacement at q_{dof}: $v_{c,max} = 8000.0\ mm$
- The number of steps $n_{tot} = 500$
- At each load step, the increment in external force is given by $\Delta\bar{\text{v}} = v_{c,max}/n_{tot}$
- The tolerance $tol = 1$
- The maximum number of iterations for each iterations of the displacement control loop $i_{\max}$

8) For displacement increment n, perform the Newton-Raphson iterations

8.1) For $i = 1$, set:

$\lambda^{n,i=1} = \lambda^{n-1}$, $\mathbf{F}_{int}^{n,i=1} = \mathbf{F}_{int}^{n-1}$, $\mathbf{K}_{structure}^{n,1} = \mathbf{K}_{structure}^{n-1}$ and $\mathbf{v}^{n,1} = \mathbf{v}^{n-1}$

8.2) Determine $\delta\mathbf{v}_r^{n,i}$and $\delta\mathbf{v}_p^{n,i}$:

$$\delta\mathbf{v}_{p,f}^{n,i} = \left(\mathbf{K}_{structure}^{n,i-1}\right)^{-1}\bar{\mathbf{F}}_{ext}$$

$$\delta\mathbf{v}_{r,f}^{n,i} = -\left(\mathbf{K}_{structure}^{n,i-1}\right)^{-1}\mathbf{F}_{unb}^{n,i-1}$$

Where the subscript f denotes the free degrees of freedom of the system

8.3) Compute the increment in the load multiplier $\delta\lambda^{n,i}$:

$$\delta\lambda^{n,i} = \begin{cases} \dfrac{\Delta\bar{\text{v}}^n}{\delta\mathbf{v}_p^{n,i}} \text{ if } i = 1\ (note,\ \delta\mathbf{v}_r^{n,1} = \mathbf{0}) \\ -\dfrac{\delta\mathbf{v}_r^{n,i}}{\delta\mathbf{v}_p^{n,i}} \text{ else} \end{cases}$$

8.4) Compute the increment in structure displacements $\Delta\mathbf{v}^{n,i}$:

$$\delta\mathbf{v}^{n,i} = \delta\lambda^{n,i}\delta\mathbf{v}_p^{n,i} + \delta\mathbf{v}_r^{n,i}$$

8.5 Update the structure displacements and the load multiplier:

$$\mathbf{v}^{n,i} = \mathbf{v}^{n,i-1} + \delta\mathbf{v}^{n,i}$$

$$\lambda^{n,i} = \lambda^{n,i-1} + \Delta\lambda^{n,i}$$

8.6) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}^{n,i}$ and $\mathbf{K}_{g,structure}^{n,i}$, as well as the structure resisting force vector $\mathbf{F}_{int}^{n,i}$. With a loop, go over all elements:

8.6.1) Determine the element displacement vector in the local reference frame $\boldsymbol{u}^{n,i}$

$$\mathbf{u}_{elem}^{n,i} = \mathbf{T}_{elem}\mathbf{v}_{elem}^{n,i}$$

Where the subscript $elem$ denotes the degrees of freedom corresponding to element $elem$

8.6.2) Using the corotational formulation for the 2d truss element derived in Question 2, compute the element displacements in the basic reference frame $\bar{\mathbf{u}} = [\bar{u}_1]^T$

8.6.3) Compute the element internal forces in the basic reference frame $\bar{\mathbf{q}}^{n,i}$:

$$\bar{\mathbf{q}}^{n,i} = \bar{\mathbf{K}}^{n,i}\bar{\mathbf{u}}^{n,i}$$

8.6.4) Determine the transformation matrix $\mathbf{L}^{n-1}$ from the basic to the local reference system:

$$\mathbf{L}^{n,i} = [-c \quad -s \quad c \quad s]$$

8.6.5) Compute the element internal force vector in the local reference frame:

$$\mathbf{Q}_{elem}^{n,i} = \left(\mathbf{L}^{n,i}\right)^T \overline{\mathbf{q}}^{n,i}$$

8.6.6) Determine the element geometric stiffness matrix in the local reference frame $\mathbf{K}_{g,elem}^{n,i}$:

$$\mathbf{K}_{geom} = \frac{\bar{q}_1}{L_n}\begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}$$

8.6.7) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}^{n,i}$ and $\mathbf{K}_{g,structure}^{n,i}$ as well as the structure internal force vector $\mathbf{F}_{int}^{n,i}$ with the element quantities

8.7) Compute the unbalanced load vector $\mathbf{F}_{unb}^{n,i} = \mathbf{F}_{int}^{n,i} - \mathbf{F}_{ext}^{n}$

8.8) Check if the Newton-Raphson procedure has converged. In the source code, convergence is achieved once

$$\left\|\boldsymbol{F}_{unb,f}^{n,i}\right\| < tol$$

8.9) If iteration i has converged, go to next load step n, else set $i = i + 1$ and go back to step **(8.2)**

The following figure shows the results obtained in this case:

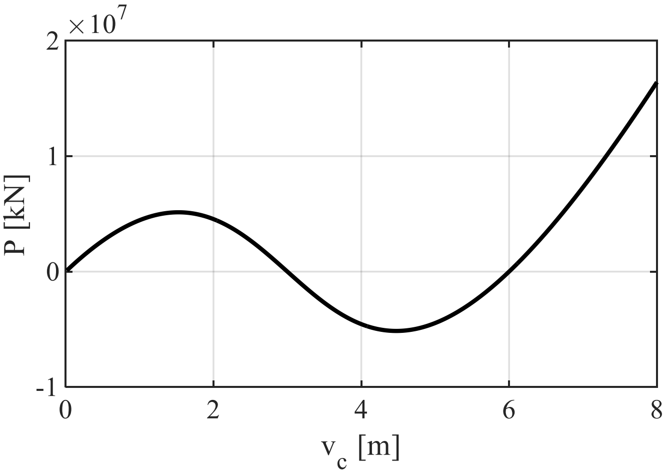


Figure 2. Vertical load-displacement relation at point c for $b_{bc} > b_{bc,min}$

Part (b)

To determine the $P - v_c$ equilibrium path using arc-length control, the following steps are used:

1) Define the member properties (E, A and l)

2) Define the connectivity matrix and the mapping matrix between local and global degrees of freedom.

Figure 1 above shows the global degrees of freedom used for the structure. The mapping matrix $\boldsymbol{numEq}$ is the same as for part (a) and is given by

$$\boldsymbol{numEq} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{bmatrix}$$

3) For each member, determine the transformation matrix **T** between local and global coordinates. In this exercise, the local x-axis is defined in the axial direction of the element

4) Assemble the initial structure stiffness matrix $\mathbf{K}_{structure}$

5) Define the boundary conditions, the fixed and the free degrees of freedom of the problem the external loads, i.e., apply the reference load $\bar{\mathbf{F}}_{ext} = [0\ 0\ 0\ 0\ \ 0 - 1\ \ 0\ 0]^T$)

6) Initialize the variables used within the arc-length control procedure

$$\lambda = 0, \mathbf{v} = \mathbf{0}$$

7) Define the parameters defining the arc-length control algorithm:

- The dof controlling the displacement $q_{dof} = 6$
- The final displacement at q_{dof}: $v_{c,max} = 8000.0\ mm$
- The number of steps $n_{tot} = 500$
- The arc-length, $\Delta\bar{l} = 20$, which is kept constant for all analysis steps
- The parameter ψ used for the arc length. Here the cylindrical arc-length is used (i.e., $\psi = 0$)
- The tolerance $tol = 1$
- The maximum number of iterations for each iterations of the arc-length control loop $i_{\max}$

8) For increment n, perform the Newton-Raphson iterations

8.1) For $i = 1$, set,

$\lambda^{n,i=1} = \lambda^{n-1}$, $\mathbf{F}_{int}^{n,i=1} = \mathbf{F}_{int}^{n-1}$, $\mathbf{K}_{structure}^{n,1} = \mathbf{K}_{structure}^{n-1}$ and $\mathbf{v}^{n,1} = \mathbf{v}^{n-1}$

8.2) Determine $\delta\mathbf{v}_r^{n,i}$ and $\delta\mathbf{v}_p^{n,i}$:

$$\delta\mathbf{v}_{p,f}^{n,i} = \left(\mathbf{K}_{structure}^{n,i-1}\right)^{-1}\bar{\mathbf{F}}_{ext}$$

$$\delta\mathbf{v}_{r,f}^{n,i} = -\left(\mathbf{K}_{structure}^{n,i-1}\right)^{-1}\mathbf{F}_{unb}^{n,i-1}$$

Where the subscript f denotes the free degrees of freedom of the system

8.3) Compute the increment in the load multiplier $\delta\lambda^{n,i}$:

$$\delta\lambda^{n,i} = \begin{cases} \pm \dfrac{\Delta\bar{l}}{\sqrt{\left(\delta\mathbf{v}_p^{n,i=1}\right)^T \delta\mathbf{v}_p^{n,i=1} + \psi^2}} \, if \; i = 1 (note, \delta\mathbf{v}_r^{n,1} = \mathbf{0}) \\ -\dfrac{\left(\delta\mathbf{v}^{n,i=1}\right)^T \delta\mathbf{v}_r^{n,i}}{(\delta\mathbf{v}^{n,i=1})^T \delta\mathbf{v}_p^{n,i} + \psi^2 \delta\lambda^{n,i=1}} \; else \end{cases}$$

The sign of the first line is taken as the one corresponding to the sign of the structure tangent stiffness matrix

8.4) Compute the increment in structure displacements $\Delta\mathbf{v}^{n,i}$:

$$\delta\mathbf{v}^{n,i} = \delta\lambda^{n,i}\delta\mathbf{v}_p^{n,i} + \delta\mathbf{v}_r^{n,i}$$

8.5) Update the structure displacements and the load multiplier:

$$\mathbf{v}^{n,i} = \mathbf{v}^{n,i-1} + \delta\mathbf{v}^{n,i}$$
$$\lambda^{n,i} = \lambda^{n,i-1} + \Delta\lambda^{n,i}$$

8.6) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}^{n,i}$ and $\mathbf{K}_{g,structure}^{n,i}$, respectively, as well as the structure resisting force vector $\mathbf{F}_{int}^{n,i}$. With a loop, go over all elements:

8.6.1) Determine the element displacement vector in the local reference frame $\boldsymbol{u}^{n,i}$

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$$\left\|\mathbf{F}_{unb,f}^{n,i}\right\| < tol$$

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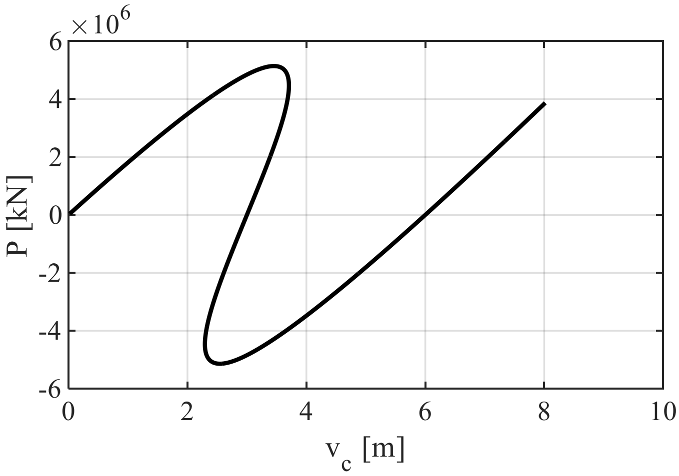

Figure 3. Vertical load-displacement relation at point c $b_{bc} < b_{bc,min}$,