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In-class Exercise Week #4: Geometric stiffness matrix

Exercise #1:

Both members in the frame below are axial force members with the following properties:

$$A_{ab} = 2mm^2, A_{bc} = 5x10^3mm^2, E = 200{,}000MPa$$

Calculate the critical load of the system for the following two cases:

1. $a = 0$
2. $a = 0.05$

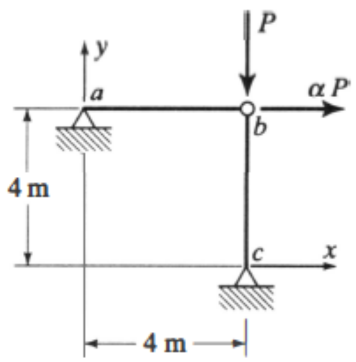


Figure 1. Planar frame under axial loading

Solution

a) For $\alpha = 0$

The submatrix of the elastic stiffness matrix of the structure, obtained by considering only the free degrees of freedom (u_b, v_b) is given by

$$\mathbf{K}_{ef} = \begin{bmatrix} \frac{EA_1}{L_1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{EA_2}{L_2} \end{bmatrix} = \frac{1}{10} \begin{array}{cc} \begin{array}{cc} u_b & v_b \end{array} & \\ \begin{bmatrix} 1 & 0 \\ 0 & 2.5 \cdot 10^3 \end{bmatrix} & \begin{array}{c} u_b \\ v_b \end{array} \end{array}$$

Similarly, the submatrix of the geometric stiffness matrix of the structure, obtained by considering only the free degrees of freedom is given by (see Slide 16 of Lecture #4):

$$\mathbf{K}_{gf} = \frac{0}{L_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{P_{ref}}{L_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -\frac{P_{ref}}{L_2} \begin{array}{cc} \begin{array}{cc} u_b & v_b \end{array} & \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{array}{c} u_b \\ v_b \end{array} \end{array}$$

Where $P_{ref} = -1$ is the external reference load.

From Slide 19 of Lecture #4, the critical load multiplier λ_{crit} is obtained by solving the following eigen value problem for the minimum eigenvalue λ

$$-\left(\mathbf{K}_{ef}\right)^{-1} \mathbf{K}_{gf} \boldsymbol{\Delta}_f = \frac{1}{\lambda} \boldsymbol{\Delta}_f$$

Where $\boldsymbol{\Delta}_f$ is an eigenvector.

Then

$$\lambda_{cr} = \min(\lambda)$$

Using the previously defined stiffness matrices $\mathbf{K}_{ef}$ and $\mathbf{K}_{gf}$, the critical load multiplier corresponds to $\lambda_{cr} = 400\ kN$.

The critical buckling load P_{cr} is the defined as

$$P_{cr} = \lambda_{cr} \cdot P_{ref} = 400\ kN$$

b) For $\alpha = 0.05$:

The solution of this part should be derived by using a displacement control algorithm; we will discuss this in Week #6 and the solution will be complemented during this week.

Exercise #2:

Member ab is a straight elastic bar with the following properties:

$$A = 1.27x10^4 mm^2, I = 3.66x10^7 mm^4, E = 200{,}000MPa$$

Calculate the critical load of the system for the following two cases:

1. $a = 0$
2. $a = 1.25x10^{-4}$

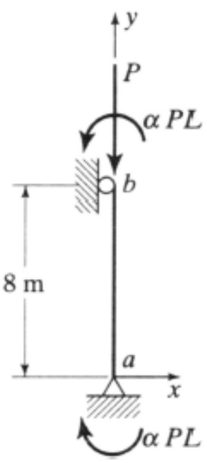
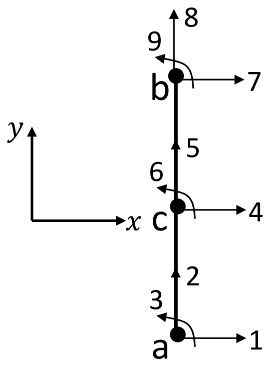


Figure 1. Member under axial and flexural loading

Solution

In this exercise, 2-dimensional (2-d) beam elements are used. The structure is divided into $n = 2$ beam elements.

The figure below shows the global degrees of freedom of the problem:



a) For $\alpha = 0$

The elastic structure stiffness matrix in the global reference system is given by

$$\mathbf{K}_e = 10^{10} \cdot \begin{bmatrix} 0.0000 & 0.0000 & -0.0003 & -0.0000 & -0.0000 & -0.0003 & 0 & 0 & 0 \\ 0.0000 & 0.0001 & 0.0000 & -0.0000 & -0.0001 & 0.0000 & 0 & 0 & 0 \\ -0.0003 & 0.0000 & 0.7320 & 0.0003 & -0.0000 & 0.3660 & 0 & 0 & 0 \\ -0.0000 & -0.0000 & 0.0003 & 0.0000 & 0.0000 & 0 & -0.0000 & -0.0000 & -0.0003 \\ -0.0000 & -0.0001 & -0.0000 & 0.0000 & 0.0001 & 0 & -0.0000 & -0.0001 & 0.0000 \\ -0.0003 & 0.0000 & 0.3660 & 0 & 0 & 1.4640 & 0.0003 & -0.0000 & 0.3660 \\ 0 & 0 & 0 & -0.0000 & -0.0000 & 0.0003 & 0.0000 & 0.0000 & 0.0003 \\ 0 & 0 & 0 & -0.0000 & -0.0001 & -0.0000 & 0.0000 & 0.0001 & -0.0000 \\ 0 & 0 & 0 & -0.0003 & 0.0000 & 0.3660 & 0.0003 & -0.0000 & 0.7320 \end{bmatrix}$$

Where the units used are [mm] and [N]

The external force vector $\mathbf{F}_{ext}$ is given by

$$\mathbf{F}_{ext} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P_{ref} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

Using Slide 16 of Lecture #4, the geometric structure stiffness matrix in the global reference frame is given by

$$\mathrm{K}_g = 10^3 \cdot \begin{bmatrix} -0.0000 & 0.0000 & 0.0001 & 0.0000 & -0.0000 & 0.0001 & 0 & 0 & 0 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0 & 0 & 0 \\ 0.0001 & -0.0000 & -0.5333 & -0.0001 & 0.0000 & 0.1333 & 0 & 0 & 0 \\ 0.0000 & -0.0000 & -0.0001 & -0.0000 & 0.0000 & 0 & 0.0000 & -0.0000 & 0.0001 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0 & -0.0000 & 0.0000 & -0.0000 \\ 0.0001 & -0.0000 & 0.1333 & 0 & 0 & -1.0667 & -0.0001 & 0.0000 & 0.1333 \\ 0 & 0 & 0 & 0.0000 & -0.0000 & -0.0001 & -0.0000 & 0.0000 & -0.0001 \\ 0 & 0 & 0 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0 & 0 & 0 & 0.0001 & -0.0000 & 0.1333 & -0.0001 & 0.0000 & -0.5333 \end{bmatrix}$$

From Slide 19 of Lecture #4, the critical load multiplier λ_{crit} is obtained by solving the following eigen value problem for the minimum eigenvalue λ

$$-\left(\mathbf{K}_{ef}\right)^{-1}\mathbf{K}_{gf}\boldsymbol{\Delta}_{f} = \frac{1}{\lambda}\boldsymbol{\Delta}_{f}$$

Where $\boldsymbol{\Delta}_{f}$ is an eigenvector.
Then

$$\lambda_{cr} = \min(\lambda)$$

Using the previously defined stiffness matrices $\mathbf{K}_{ef}$ and $\mathbf{K}_{gf}$, the critical load multiplier corresponds to $\lambda_{cr} = 1137\ kN$.

The critical buckling load P_{cr} is the defined as

$$P_{cr} = \lambda_{cr} \cdot P_{ref} = 1137\ kN$$

This case corresponds to the Euler buckling load (see the Example on Slide 21 of Lecture #4).

The theoretical value based on the Euler buckling theory is as follows,

$$P_{cr,th} = 1129\ kN$$

By increasing the number of beam elements to idealize the column, the predicted P_{cr} should converge to the theoretical value. In this case, after using four elements, $P_{cr} = 1129\ kN$.

b) For $\alpha = 1.25 \cdot 10^{-4}$:

The solution of this part should be derived by using a displacement control algorithm; we will discuss this in Week #6 and the solution will be complemented during this week.