FACULTE ENVIRONNEMENT NATUREL, ARCHITECTURAL ET CONSTRUIT
ENAC
INSTITUT D'INGÉNIERIE CIVILE IIC
Laboratoire des Structures Métalliques Résilientes
RESSLAB

Téléphone : +41 21 693 24 27
Fax : +41 21 693 28 68
E-mail : dimitrios.lignos@epfl.ch
Site web : http://resslab.epfl.ch
Address: EPFL ENAC IIC RESSLAB
GC B3 485, Station 18,
CH-1015, Lausanne

In-class Exercise Week #3: Truss, frame and zero length elements

Exercise #1:

What is the magnitude and direction of the force P at point a required to displace that point a vertically downward 5mm without any horizontal displacement? Assume that $E = 200\,GPa$; the rotational stiffness $k_R = 1000\frac{kNmm}{rad}$; the translational stiffness $k_T = 50kN/mm$.

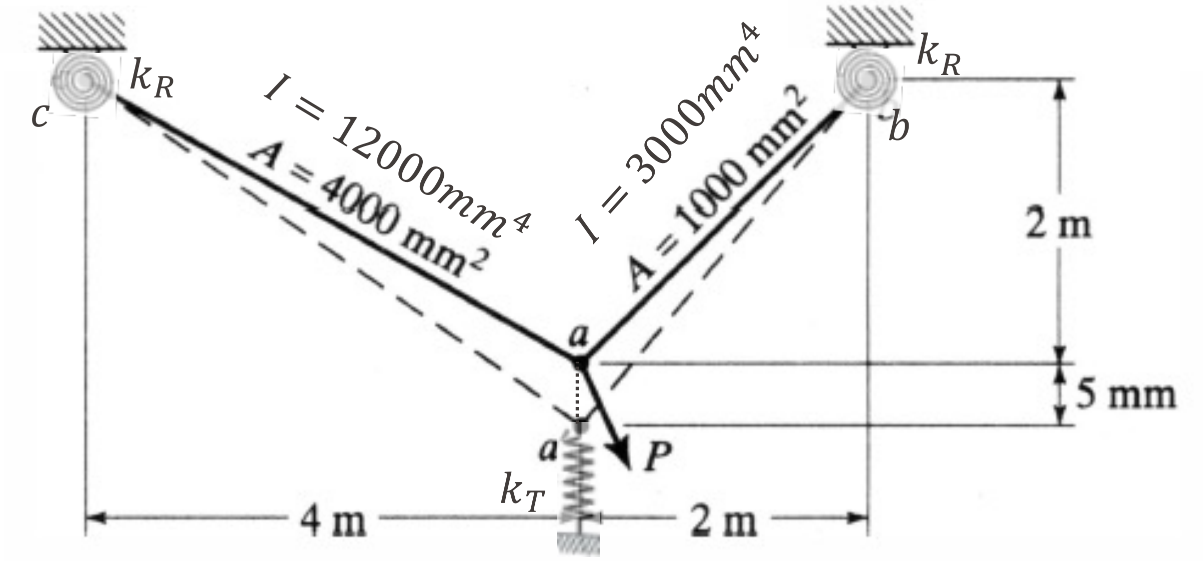
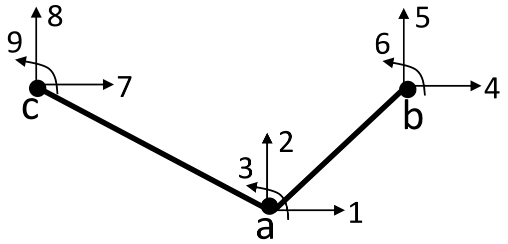


Figure 1. Schematic representation of the planar frame when subjected to the force P

Solution:
The procedure described in Exercise 2 may be used to solve the problem, considering the following updates:

1. 2d elastic beam elements are used instead of truss elements
2. The stiffness of each spring element should be added to the global stiffness matrix according to the appropriate global degree of freedom corresponding to the spring.

The following figure shows the global degrees of freedom of the problem:



The structure stiffness matrix considering only the two beam-column elements is as follows:

$$\boldsymbol{K}_{\text{structure}} = 10^6 \cdot \begin{bmatrix} 0.1785 & -0.0362 & -0.0006 & -0.0354 & -0.0354 & -0.0003 & -0.1431 & 0.0716 & -0.0003 \\ -0.0362 & 0.0711 & -0.0003 & -0.0354 & -0.0354 & 0.0003 & 0.0716 & -0.0358 & -0.0006 \\ -0.0006 & -0.0003 & 2.9952 & 0.0003 & -0.0003 & 0.4243 & 0.0003 & 0.0006 & 1.0733 \\ -0.0354 & -0.0354 & 0.0003 & 0.0354 & 0.0354 & 0.0003 & 0 & 0 & 0 \\ -0.0354 & -0.0354 & -0.0003 & 0.0354 & 0.0354 & -0.0003 & 0 & 0 & 0 \\ -0.0003 & 0.0003 & 0.4243 & 0.0003 & -0.0003 & 0.8485 & 0 & 0 & 0 \\ -0.1431 & 0.0716 & 0.0003 & 0 & 0 & 0 & 0.1431 & -0.0716 & 0.0003 \\ 0.0716 & -0.0358 & 0.0006 & 0 & 0 & 0 & -0.0716 & 0.0358 & 0.0006 \\ -0.0003 & -0.0006 & 1.0733 & 0 & 0 & 0 & 0.0003 & 0.0006 & 2.1466 \end{bmatrix}$$

Where the units used are [mm] and [N]
NOTE: That the matrix is symmetric

The stiffness of each spring is then incorporated into the structure stiffness matrix at the appropriate global degrees of freedom:

- The stiffness of the vertical spring is added to the structure stiffness matrix at the stiffness index (2,2)
- The stiffness of the rotational springs is added to the structure stiffness matrix at the stiffness indices (6,6) and (9,9)

The structure stiffness matrix is therefore given as follows:

$$\boldsymbol{K}_{\text{structure}} = 10^6 \cdot \begin{bmatrix} 0.1785 & -0.0362 & -0.0006 & -0.0354 & -0.0354 & -0.0003 & -0.1431 & 0.0716 & -0.0003 \\ -0.0362 & 0.0711 + k_T & -0.0003 & -0.0354 & -0.0354 & 0.0003 & 0.0716 & -0.0358 & -0.0006 \\ -0.0006 & -0.0003 & 2.9952 & 0.0003 & -0.0003 & 0.4243 & 0.0003 & 0.0006 & 1.0733 \\ -0.0354 & -0.0354 & 0.0003 & 0.0354 & 0.0354 & 0.0003 & 0 & 0 & 0 \\ -0.0354 & -0.0354 & -0.0003 & 0.0354 & 0.0354 & -0.0003 & 0 & 0 & 0 \\ -0.0003 & 0.0003 & 0.4243 & 0.0003 & -0.0003 & 0.8485 + k_R & 0 & 0 & 0 \\ -0.1431 & 0.0716 & 0.0003 & 0 & 0 & 0 & 0.1431 & -0.0716 & 0.0003 \\ 0.0716 & -0.0358 & 0.0006 & 0 & 0 & 0 & -0.0716 & 0.0358 & 0.0006 \\ -0.0003 & -0.0006 & 1.0733 & 0 & 0 & 0 & 0.0003 & 0.0006 & 2.14660 + k_R \end{bmatrix}$$

The structure displacement vector in the global reference system **v** is given by

$$\mathbf{v} = \begin{pmatrix} 0 \\ -5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The force components F_x and F_y are obtained by solving the global force-displacement relation given by

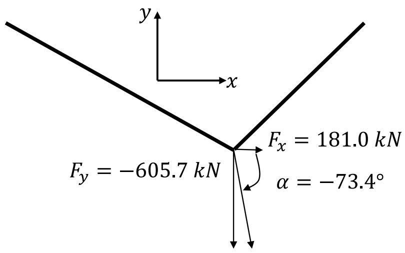
$$\mathbf{F} = \mathbf{k}_{\text{structure}} \cdot \mathbf{v}$$

The components F_x and F_y correspond to $\mathbf{F}(1)$ and $\mathbf{F}(2)$, respectively.

The following components are obtained after solving the system:

$F_x = 181.0\ kN$ and $F_y = -605.7\ kN$.

The force is therefore oriented at angle $\alpha = -73.4°$.



Exercise #2:

Using the quartic polynomial, $v = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4$, construct the shape function for the illustrated flexural element.

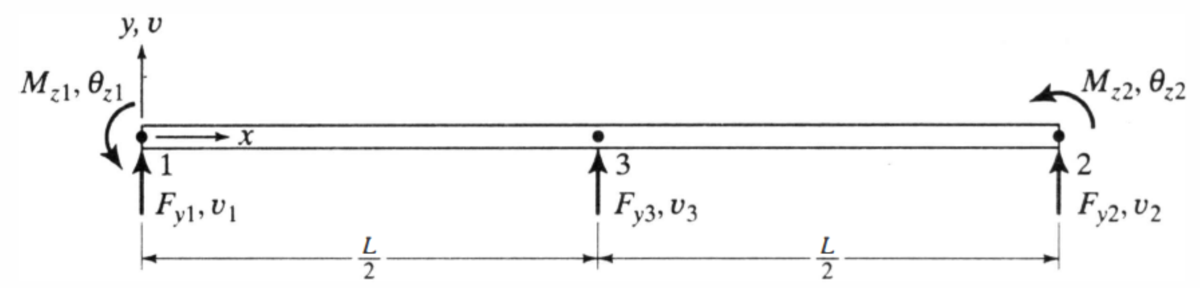


Figure 2. Flexural element

Solution

The rotation θ_z is obtained from the displacement v using

$$\frac{dv}{dx} = \theta_z = a_2 + 2a_3x + 3a_4x^2 + 4a_5x^3 \tag{1.1}$$

Evaluating the function $v(x)$ at nodes 1, 2 and 3 gives

$$\begin{cases} v_1 = v(x=0) = a_1 \\ v_2 = v(x=L) = a_1 + a_2L + a_3L^2 + a_4L^3 + a_5L^4 \\ v_3 = v\left(x=\frac{L}{2}\right) = a_1 + a_2\frac{L}{2} + a_3\frac{L^2}{4} + a_4\frac{L^3}{8} + a_5\frac{L^4}{16} \end{cases} \tag{1.2}$$

Similarly, evaluating the function $\theta_z(x)$ at nodes 1and 2 gives

$$\begin{cases} \theta_{z,1} = \theta_z(x=0) = a_2 \\ \theta_{z,2} = \theta_z(x=L) = a_2 + 2a_3L + 3a_4L^2 + 4a_5L^3 \end{cases} \tag{1.3}$$

Using Equations (1.2) and (1.3), the following system of equations should be solved in order to determine the remaining unknows a_3, a_4 and a_5

$$\begin{cases} v_2 = v_1 + \theta_{z,1}L + a_3L^2 + a_4L^3 + a_5L^4 \\ v_3 = v_1 + \theta_{z,1}\frac{L}{2} + a_3\frac{L^2}{4} + a_4\frac{L^3}{8} + a_5\frac{L^4}{16} \\ \theta_{z,2} = \theta_{z,1} + 2a_3L + 3a_4L^2 + 4a_5L^3 \end{cases} \tag{1.4}$$

Solving this system of equations for a_3, a_4 and a_5 gives

$$\begin{cases} a_3 = \frac{1}{L^2}\left(-11v_1 - 5v_2 + 16v_3 - 4L\theta_{z,1} + L\theta_{z,2}\right) \\ a_4 = \frac{1}{L^3}\left(18v_1 + 14v_2 - 32v_3 + 5L\theta_{z,1} - 3L\theta_{z,2}\right) \\ a_5 = \frac{1}{L^4}\left(-8v_1 - 8v_2 + 16v_3 - 2L\theta_{z,1} + 2L\theta_{z,2}\right) \end{cases} \tag{1.5}$$

Finally, the shape function v is given by

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4\theta_{z,1} + N_5\theta_{z,2} \tag{1.6}$$

With

$$\begin{cases} N_1 = 1 - 11\left(\frac{x}{L}\right)^2 + 18\left(\frac{x}{L}\right)^3 - 8\left(\frac{x}{L}\right)^4 \\ N_2 = -5\left(\frac{x}{L}\right)^2 + 14\left(\frac{x}{L}\right)^3 - 8\left(\frac{x}{L}\right)^4 \\ N_3 = 16\left(\frac{x}{L}\right)^2 - 32\left(\frac{x}{L}\right)^3 + 16\left(\frac{x}{L}\right)^4 \\ N_4 = x - 4L\left(\frac{x}{L}\right)^2 + 5L\left(\frac{x}{L}\right)^3 - 2L\left(\frac{x}{L}\right)^4 \\ N_5 = L\left(\frac{x}{L}\right)^2 - 3L\left(\frac{x}{L}\right)^3 + 2L\left(\frac{x}{L}\right)^4 \end{cases} \tag{1.7}$$

Exercise #3:

Formulate the stiffness matrix for a four-node torsional beam-column element (Young's modulus, E and shear modulus, G), using for the rotational displacement θ_x the following shape function,

$$\theta_x = N_1\theta_{x1} + N_2\theta_{x2} + N_3\theta_{x3} + N_4\theta_{x4}$$

Where,

$$N_1 = \frac{(L-x)(2L-x)(3L-x)}{6L^3} \qquad N_3 = \frac{x(L-x)(x-3L)}{2L^3}$$

$$N_2 = \frac{x(2L-x)(3L-x)}{2L^3} \qquad N_4 = \frac{x(L-x)(2L-x)}{6L^3}$$

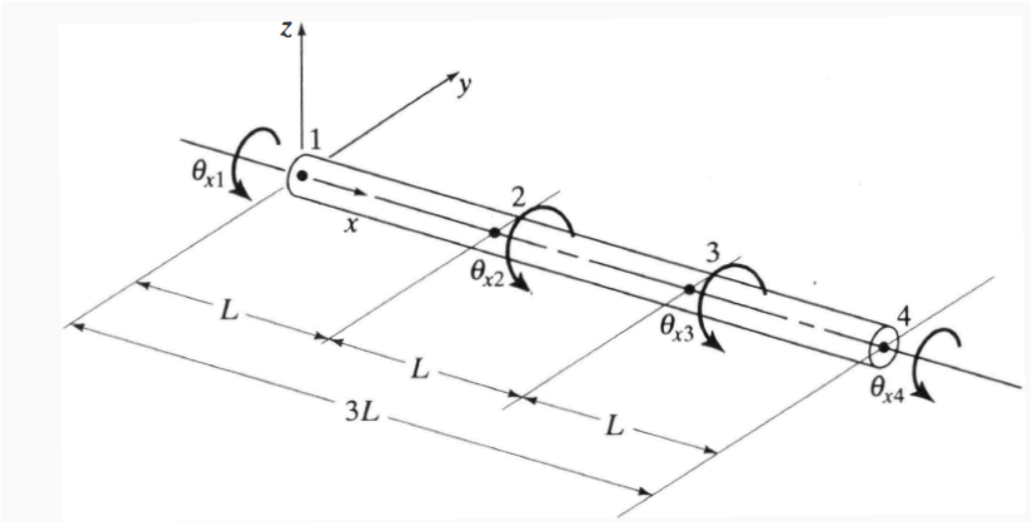


Figure 2. Beam-column under torsion

It can be proven that the stiffness matrix of a uniform beam-column element of length L under uniform torsion is as follows:

$$[k] = \int_0^L \{N'\}G\lfloor N'\rfloor J dx$$

Where J is the torsional constant of the cross section

Solution:
First, the vector N' is computed

$$N' = \begin{pmatrix} N_1{}' \\ N_2' \\ N_3' \\ N_4{}' \end{pmatrix} = \begin{pmatrix} \frac{2x}{L^2} - \frac{11}{6L} - \frac{x^2}{2L^3} \\ \frac{3}{L} - \frac{5x}{L^2} + \frac{3x^2}{2L^3} \\ \frac{4x}{L^2} - \frac{3}{2L} - \frac{3x^2}{2L^3} \\ \frac{1}{3L} - \frac{x}{L^2} + \frac{x^2}{2L^3} \end{pmatrix} \tag{2.1}$$

Using the hint, each component of the stiffness matrix K is computed using

$$K_{ij} = GJ \int_0^{3L} N_i' \cdot N_j' dx \tag{2.2}$$

From this equation, it is evident that the matrix K is symmetric (i.e., $K_{ij} = K_{ji}$).
Computing a few components of K gives

$$\begin{aligned} K_{11} &= GJ \int_0^{3L} N_1' \cdot N_1' dx \\ &= GJ \int_0^{3L} \frac{121}{36L^2} - \frac{22x}{3L^3} + \frac{35x^2}{6L^4} - \frac{2x^3}{L^5} + \frac{x^4}{4L^6} dx \\ &= GJ \left[\frac{121x}{36L^2} - \frac{11x^2}{3L^3} + \frac{35x^3}{18L^4} - \frac{2x^4}{4L^5} + \frac{x^5}{20L^6} \right]_0^{3L} \\ &= \frac{37}{30L} \end{aligned}$$

And

$$\begin{aligned} K_{34} &= GJ \int_0^{3L} N_3' \cdot N_4' dx \\ &= GJ \int_0^{3L} \frac{17x}{6L^3} - \frac{1}{2L^2} - \frac{21x^2}{4L^4} + \frac{7x^3}{2L^5} - \frac{3x^4}{4L^6} dx \\ &= GJ \left[\frac{17x^2}{3L^3} - \frac{x}{2L^2} - \frac{21x^3}{12L^4} + \frac{7x^4}{8L^5} - \frac{3x^5}{20L^6} \right]_0^{3L} \\ &= -\frac{63}{40L} \end{aligned}$$

Repeating the same operation for all indices i and j yields the following stiffness matrix

$$K = \frac{GJ}{12L} \begin{bmatrix} 14.8 & & (Sym) & \\ -18.9 & 43.2 & & \\ 5.4 & -29.7 & 43.2 & \\ -1.3 & 5.4 & -18.9 & 14.8 \end{bmatrix}$$