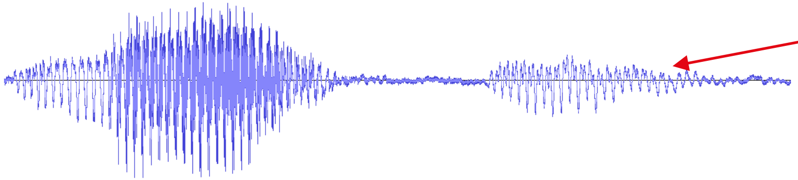
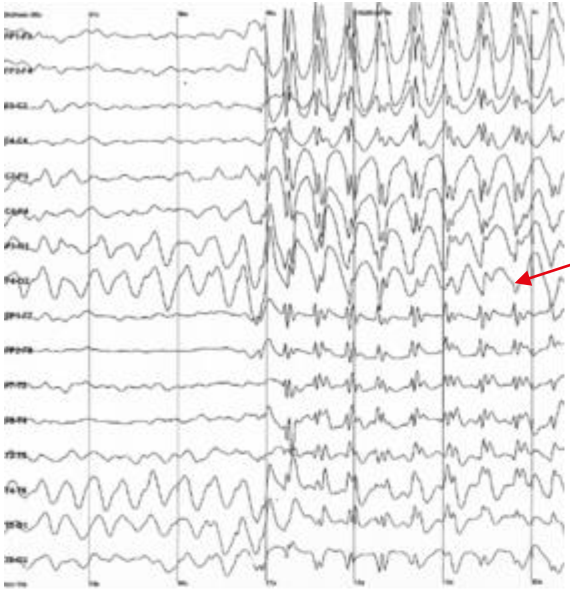
Introduction to
The Fundamental Concepts of Signal Processing

Why signal processing

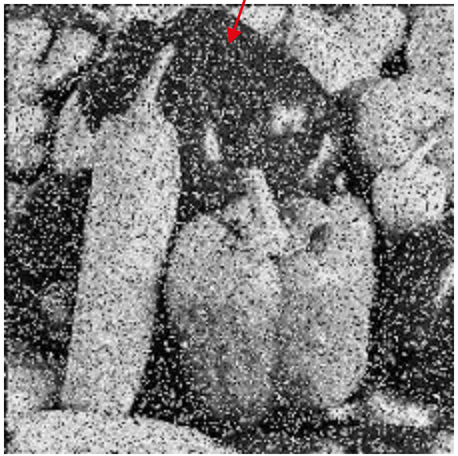


Univariate
speech signal

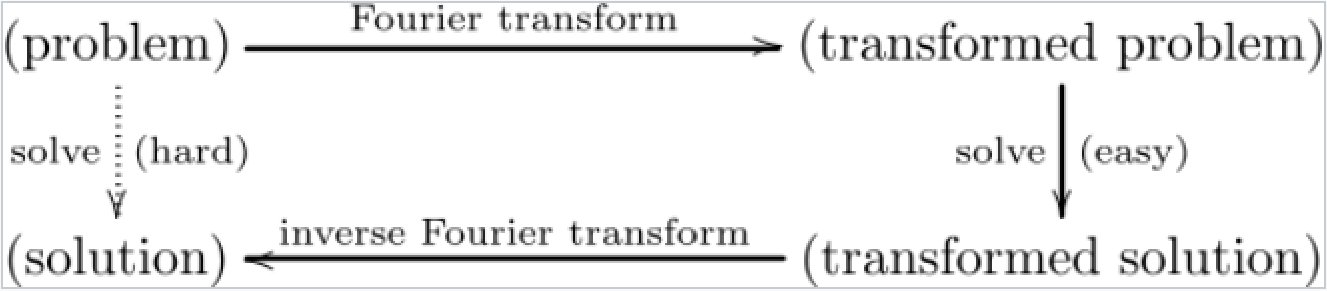
Bidimensional
image



Multivariate
EEG signal



Why signal processing



Some problems, such as certain differential equations, become easier to solve when the Fourier transform is applied. In that case the solution to the original problem is recovered using the inverse Fourier transform.

*picture from wikipedia

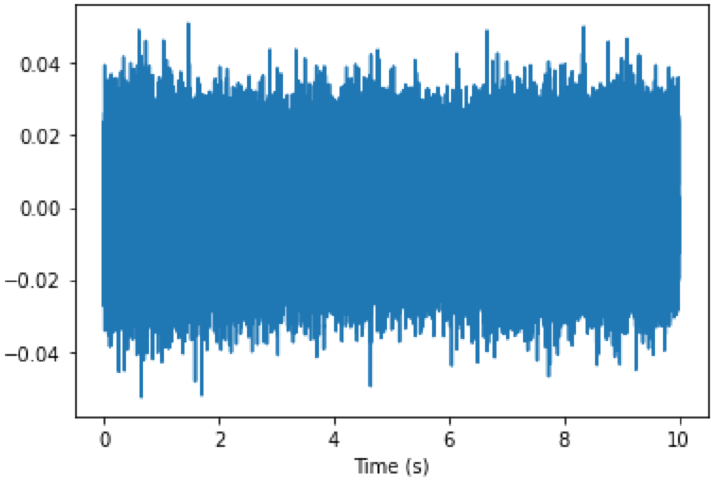
I – Fourier Transform

Fourier Transform

Considering a signal s:

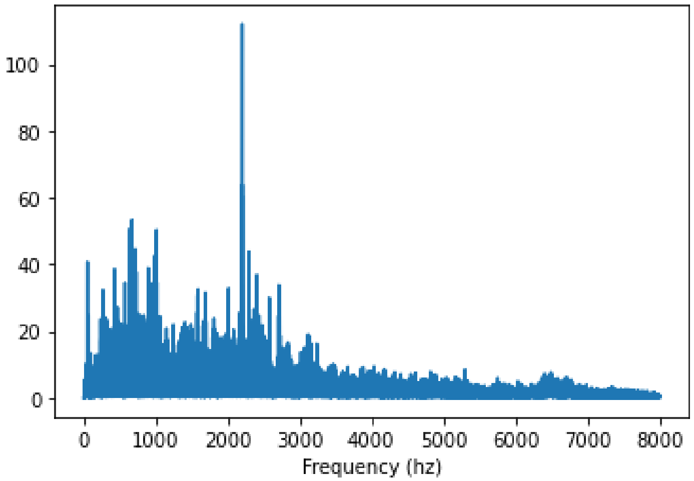
$$TF[s]_{(f)} = \int_{-\infty}^{+\infty} s(t)e^{-2i\pi f t}dt$$

Input signal



TF

Module of the fourier transform

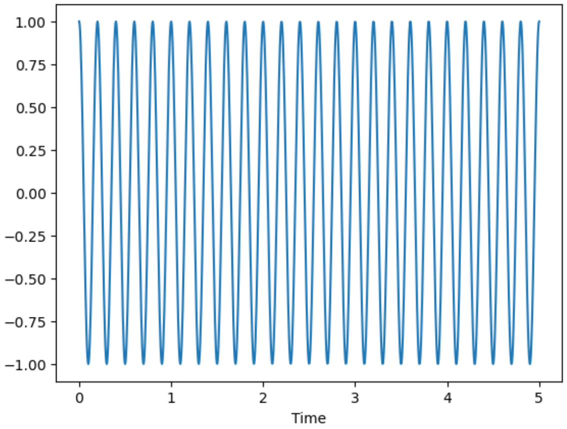


Introduction to the fundamental concepts of signal processing

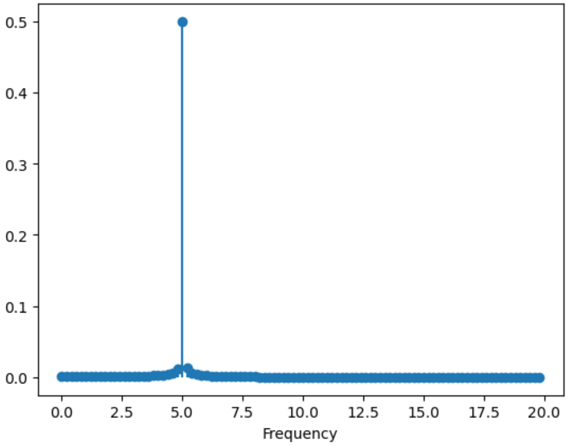
Fourier Transform

https://www.geogebra.org/m/t9uspumz#material/aenhzjpz

Time signal



FFT spectrum



Fourier Transform

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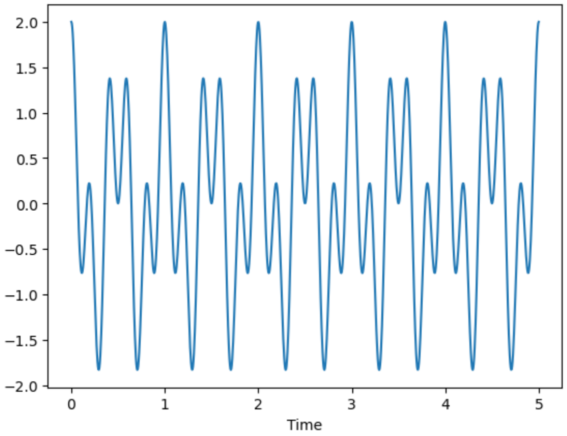
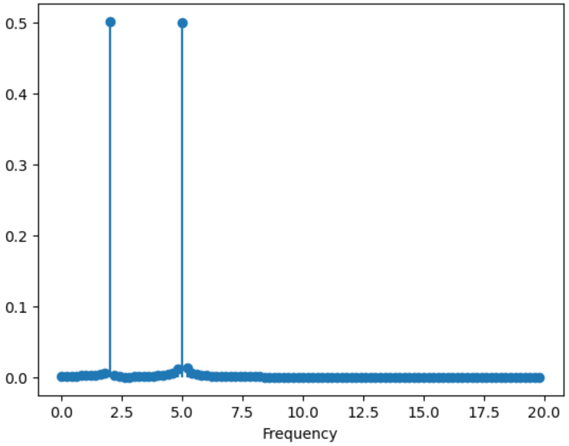
Time signal



FFT spectrum



Fourier Transform

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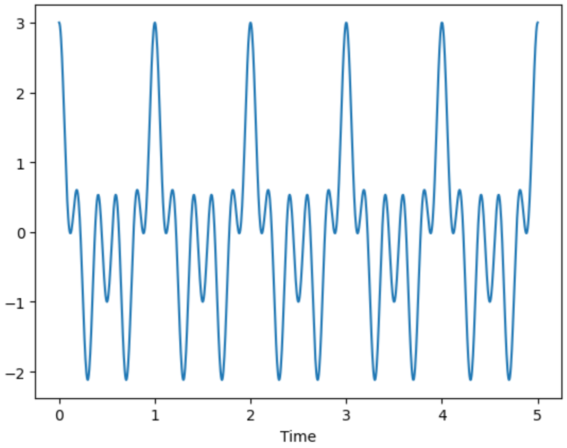
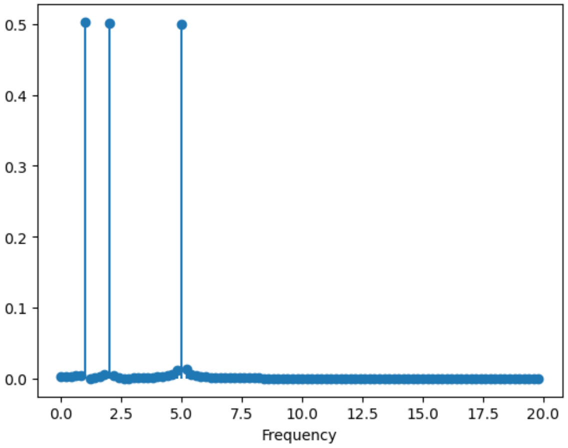
Time signal



FFT spectrum



Fourier Transform

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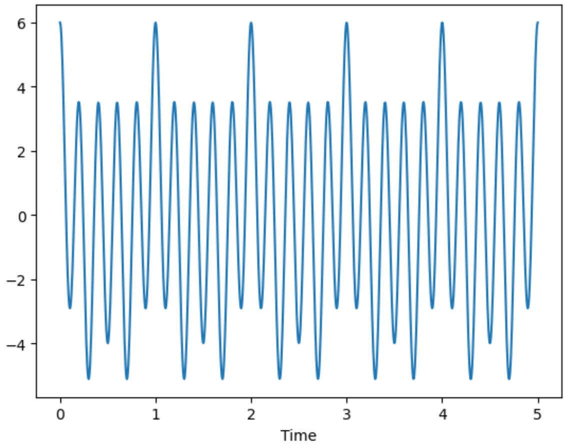
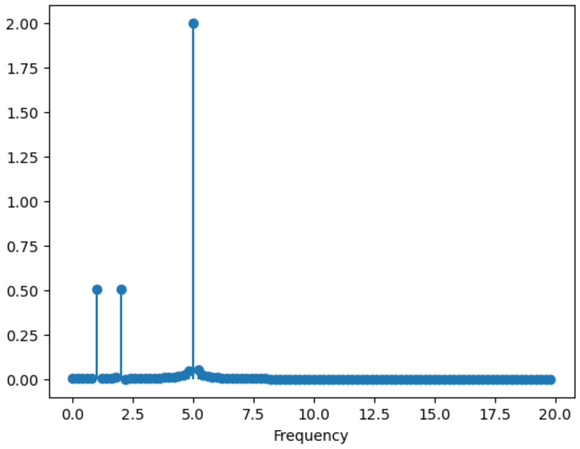
Time signal



FFT spectrum



Fourier Transform

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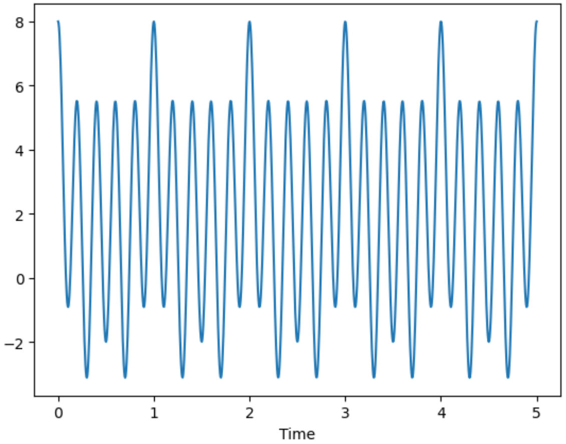
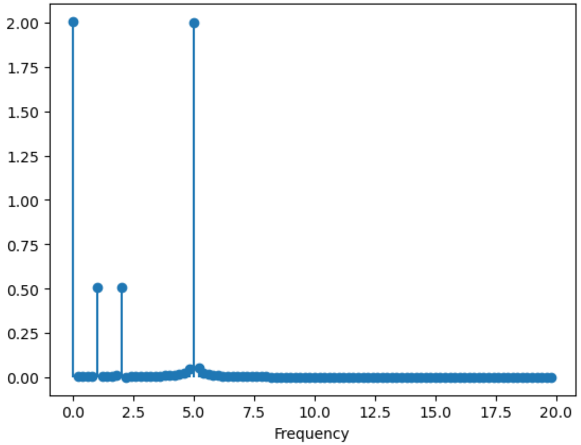
Time signal



FFT spectrum



Fourier Transform

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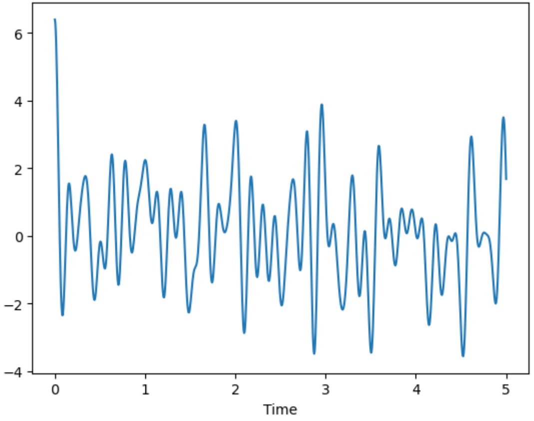
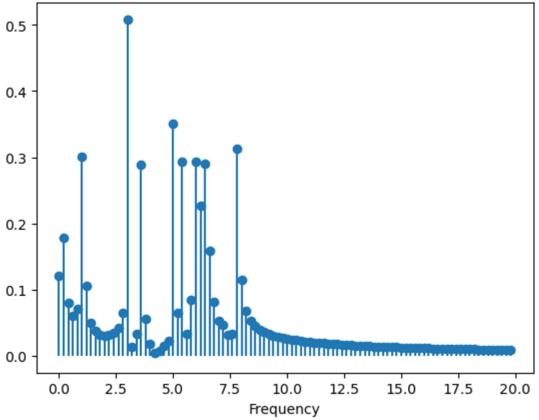
Time signal



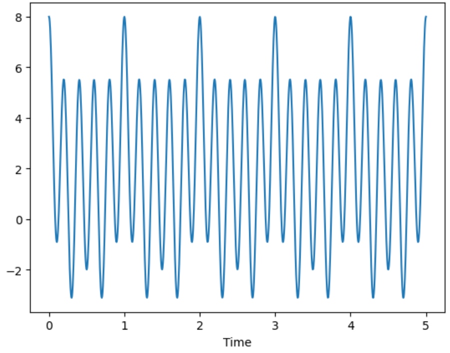
FFT spectrum

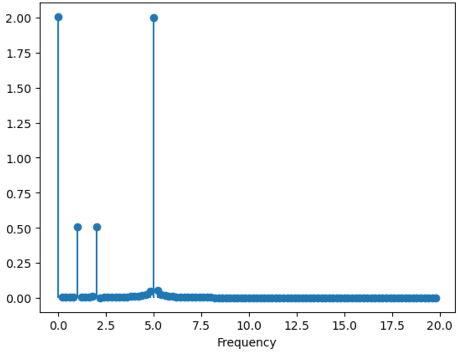


Periodicity in Fourier Transform

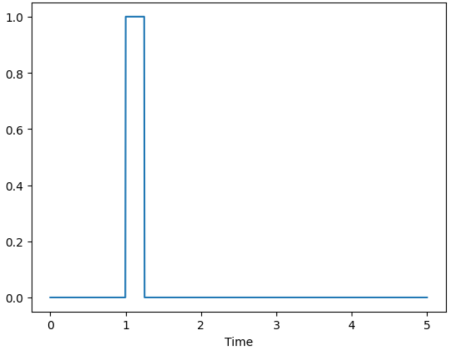
Time signal FFT spectrum

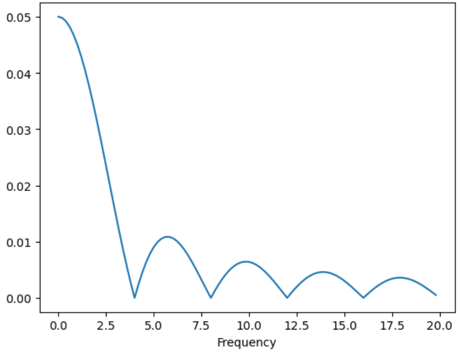
If one is periodic, then the other is discrete





If one not periodic, the other is continuous

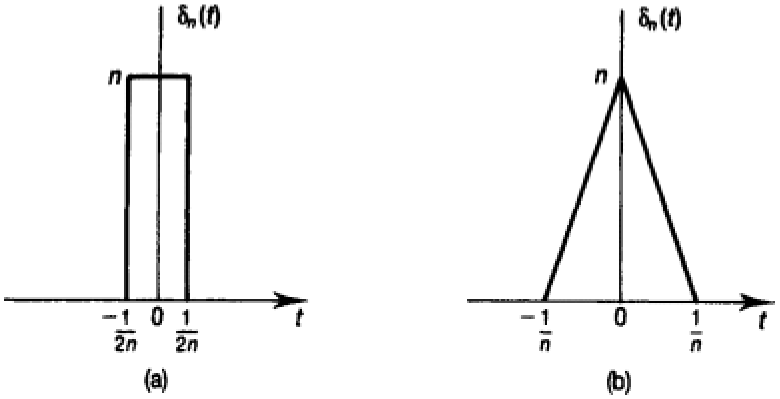




If one has finite support, the other has infinite support

Dirac

$$\delta_{f_c} = 0 \qquad \forall f \neq f_c$$
$$= \underline{\textbf{Undefined}} \qquad \text{if } f = f_c$$



The dirac function

$$\delta_{f_c} = 0 \qquad \forall f \neq f_c$$
$$= \textbf{Undefined} \qquad \text{if } f = f_c$$

Note 1 : Integrate a dirac alone is equal to 1

$$. \int_{-\infty}^{+\infty} \delta_{f_c} df = \int_{-\infty}^{+\infty} \delta_{f_c} \, 1 \, df = 1$$

Note 2 : Integrate can be «associated» to a real or complex number

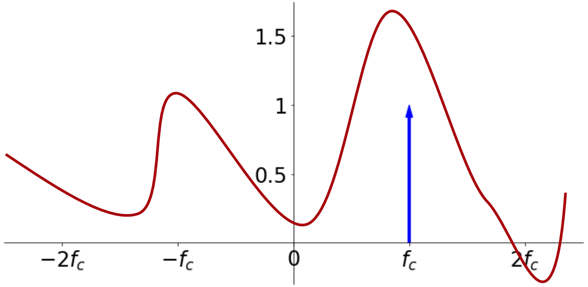
$$. \int_{-\infty}^{+\infty} (\frac{1}{2}\delta_{f_c}) K(f) df = \frac{1}{2}\int_{-\infty}^{+\infty} \delta_{f_c} K(f) df = \frac{1}{2} K(f_c)$$

The dirac function

$$\delta_{f_c} = 0 \qquad \forall f \neq f_c$$
$$= \textbf{Undefined} \qquad \text{if } f = f_c$$

Dirac is not a function but a mathematical object that has the following property:

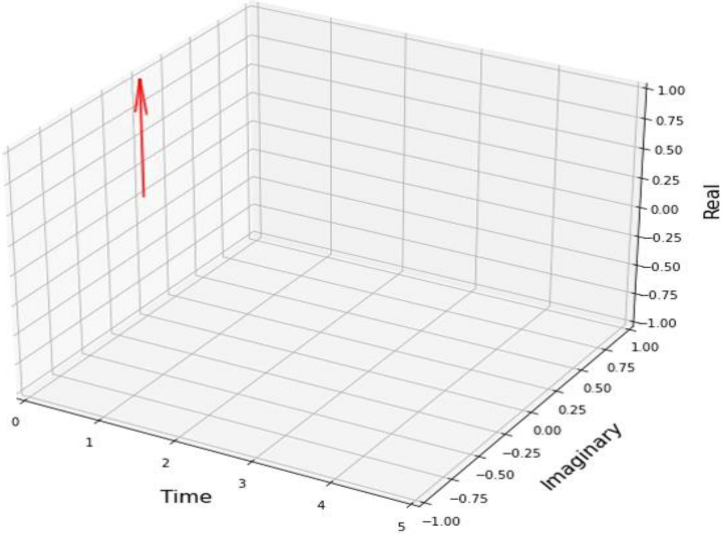
$$\int_{-\infty}^{+\infty} \delta_{f_c} K(f)df = K(f_c)$$

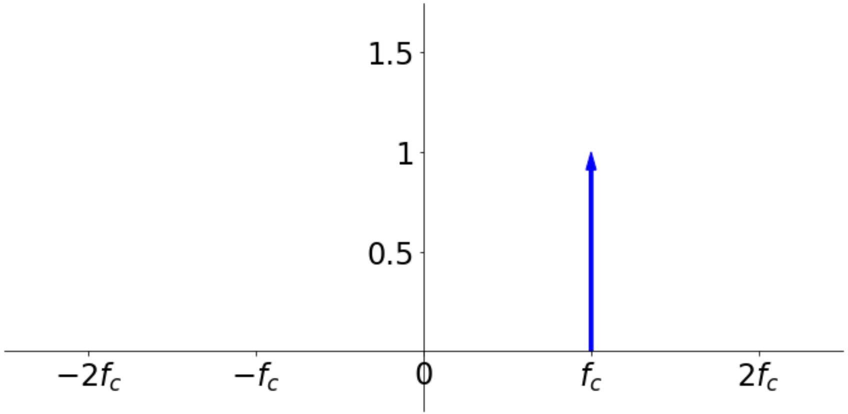


Fundamental notions of Fourier Transform

Complex exponential vs Dirac function

(time) $e^{2i\pi f_c t}$ $\xrightarrow{\text{TF}}$ δ_{f_c} (frequency)

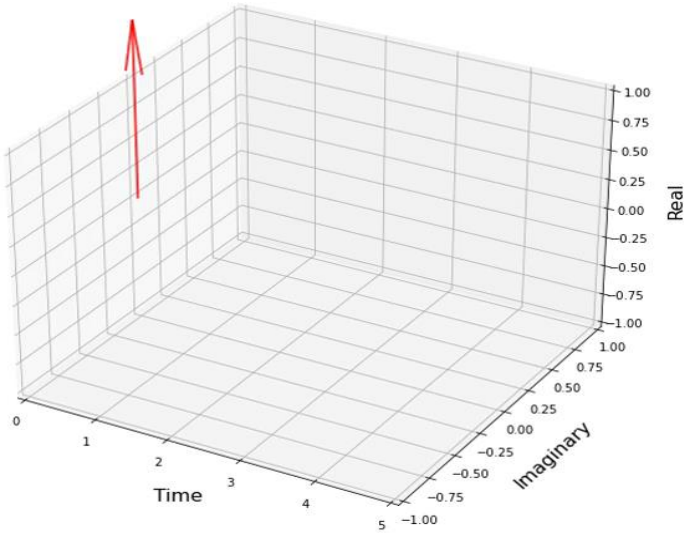


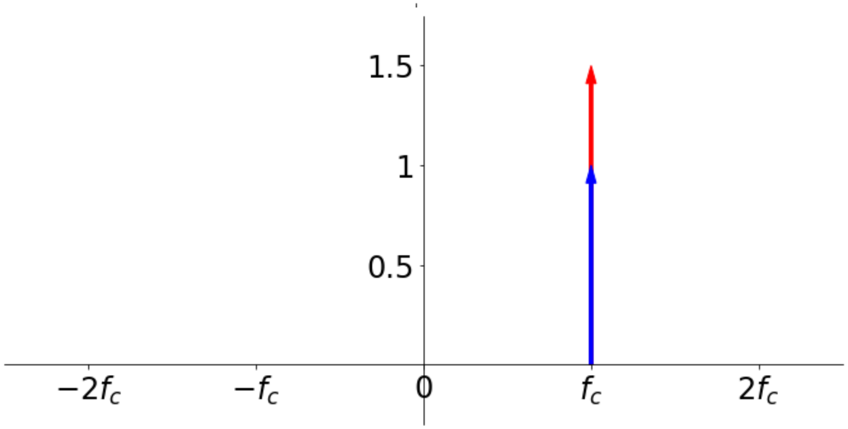


Concepts of amplitude, phase and frequency

- The amplitude: radius of the circle

(time) $1.5e^{2i\pi f_c t}$ $\xrightarrow{\text{TF}}$ $1.5\delta_{f_c}$ (frequency)

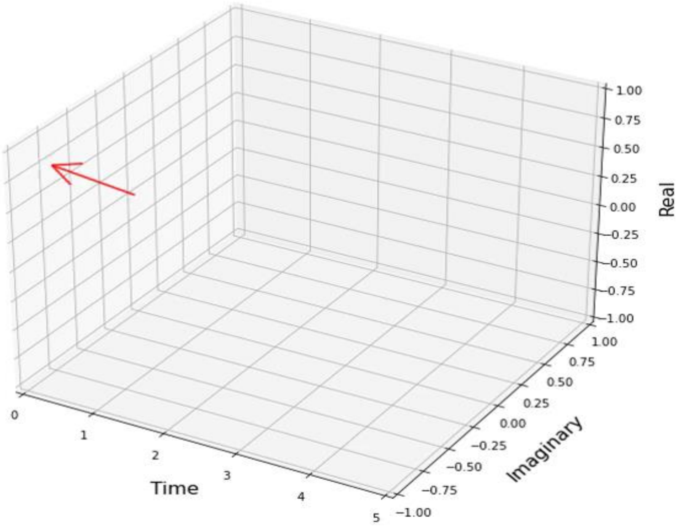


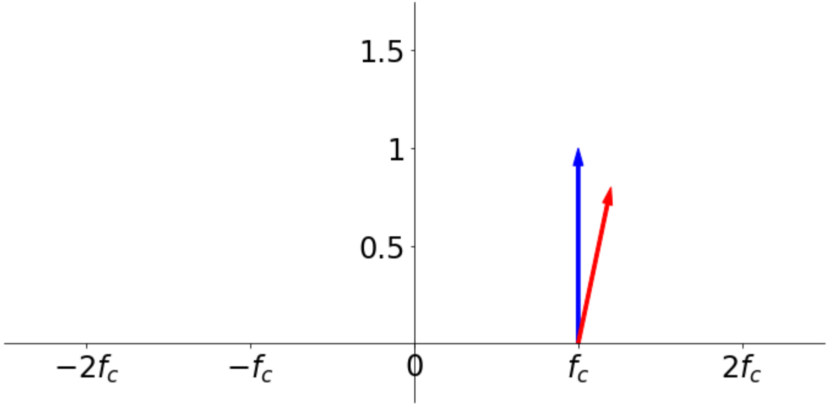


Concepts of amplitude, phase and frequency

- The phase: position at the origine or delay

(time) $e^{2i\pi(f_c t - \frac{1}{8})}$ $\xrightarrow{\text{TF}}$ $e^{-i\pi/4}\delta_{f_c}$ (frequency)

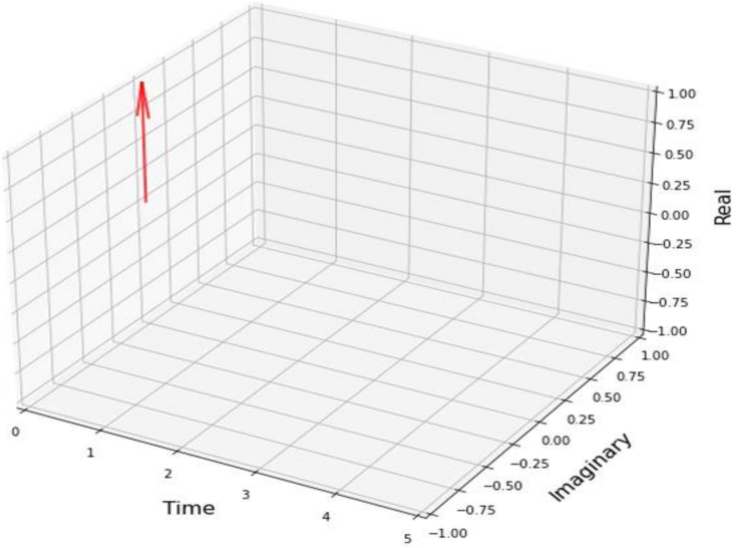


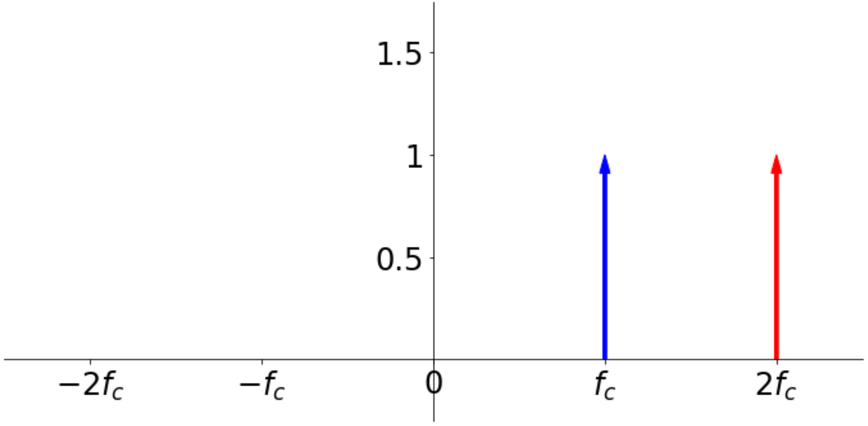


Concepts of amplitude, phase and frequency

- The frequency: it corresponds to the angular speed

(time) $e^{2i\pi(2f_c t)}$ $\xrightarrow{\text{TF}}$ δ_{2f_c} (frequency)

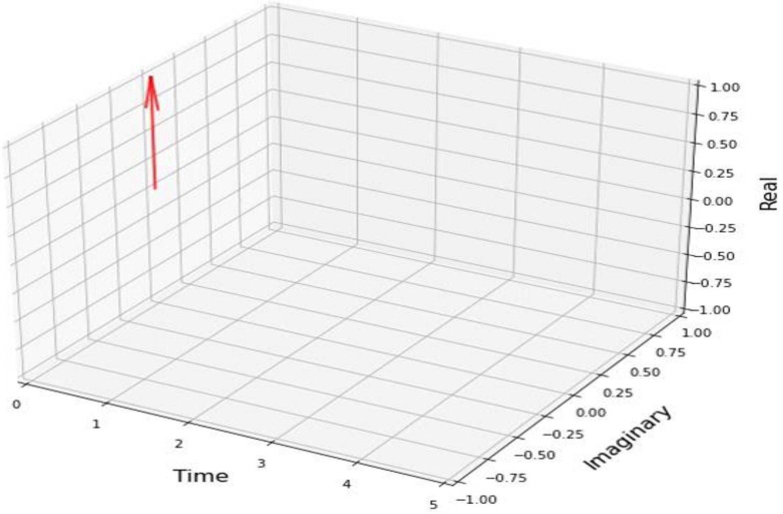


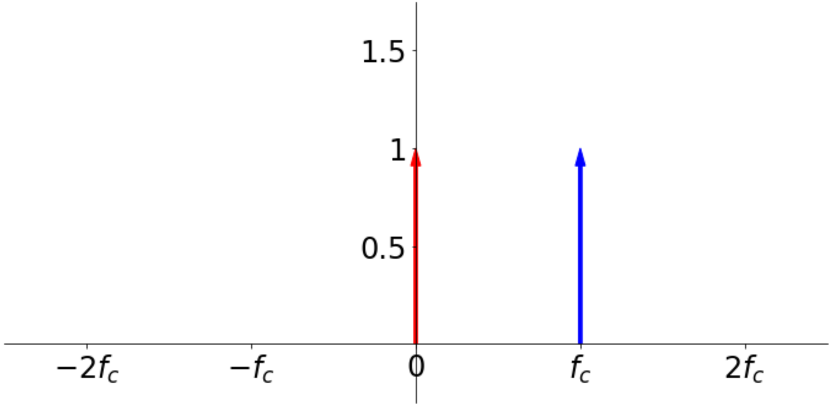


Concepts of amplitude, phase and frequency

- The frequency: it corresponds to the angular speed. Null speed:

(time) $e^{2i\pi(0 f_c t)} = 1$ $\xrightarrow{\text{TF}}$ δ_0 (frequency)

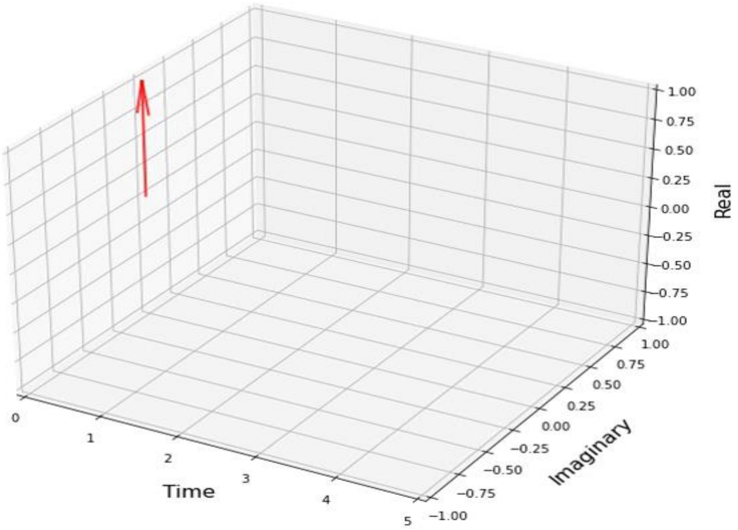


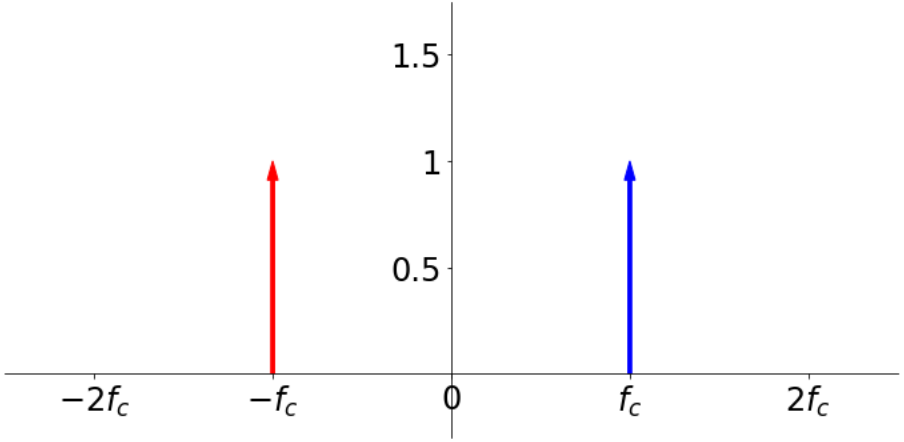


Concepts of amplitude, phase and frequency

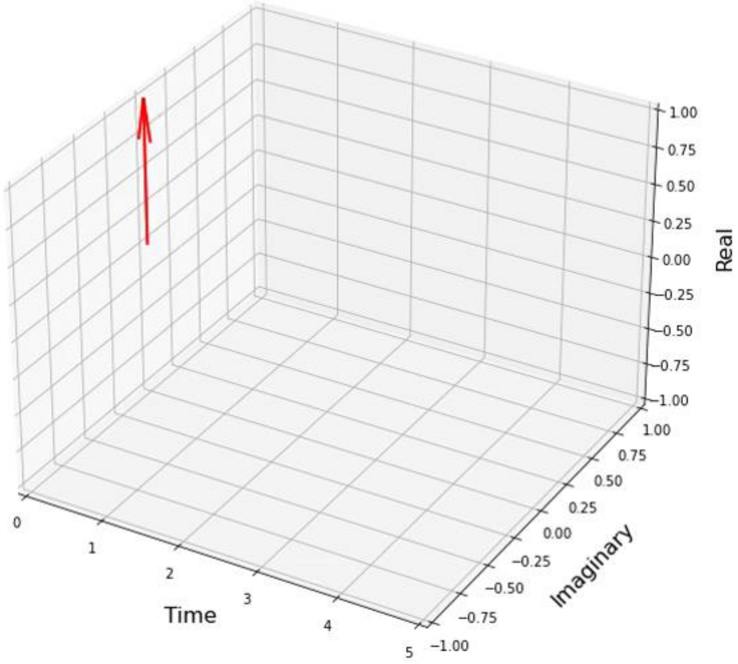
- The negative frequency: rotation in the opposite direction

(time) $e^{-2i\pi(2f_c t)}$ $\xrightarrow{\text{TF}}$ δ_{-f_c} (frequency)

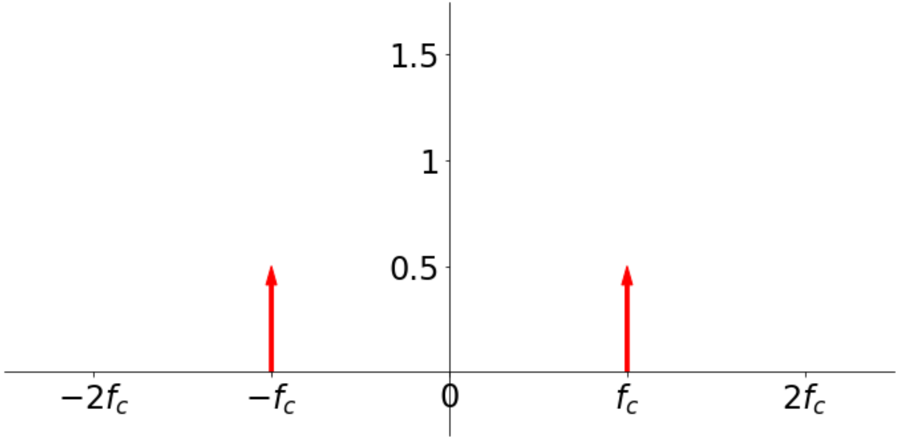




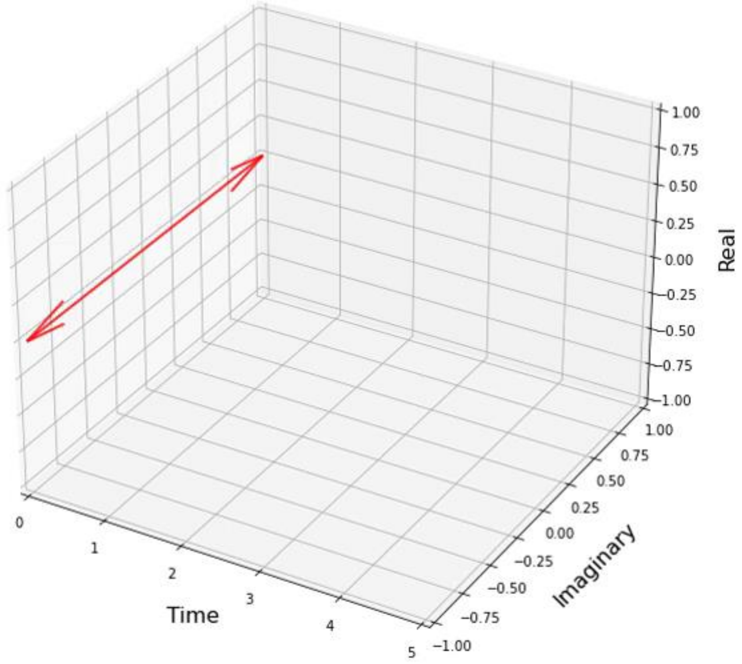
Understand the Fourier Transform of the cosinus



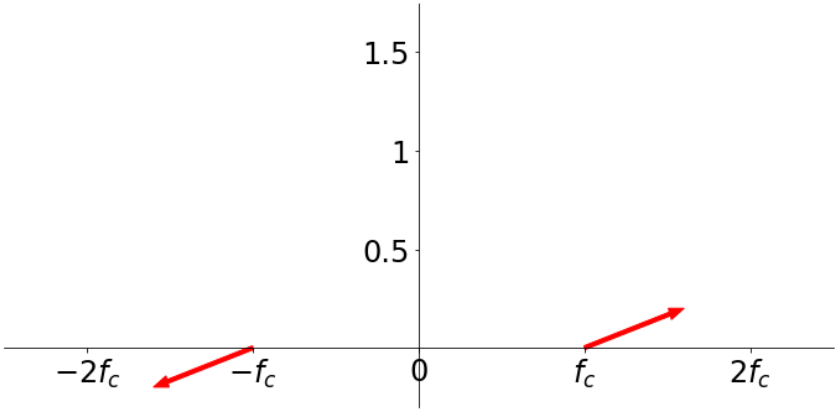
TF[cos($2\pi f_c t$)] = $1/2(\delta_{-f_c} + \delta_{f_c})$



Understand the Fourier Transform of the sinus



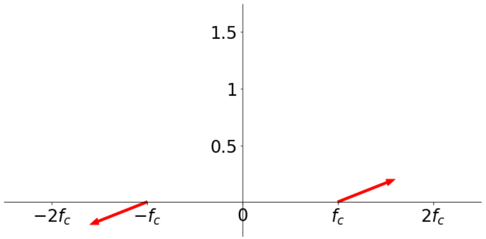
TF[sin($2\pi f_c t$)] = $-i/2(\delta_{-f_c} - \delta_{f_c})$



Additional comments

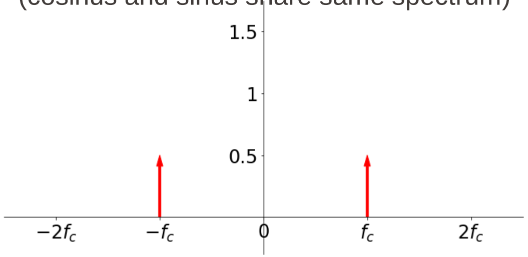
1. Be carefull: Fourier transform is different from a spectrum (modulus of the FT)

FT of the sinus function

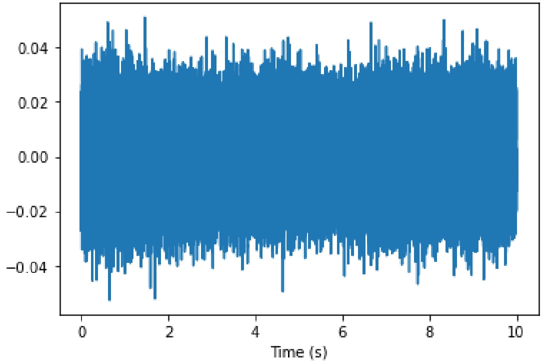


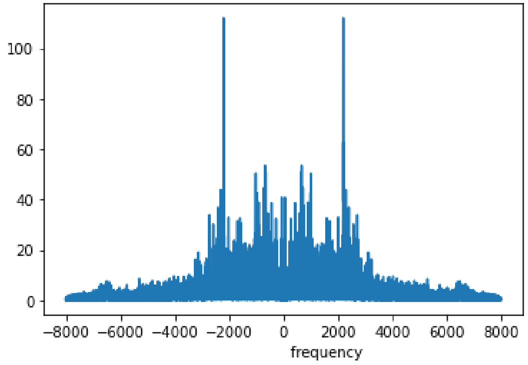
Spectrum of the sinus function

(cosinus and sinus share same spectrum)



2. The spectrum of a real function is symmetric about 0





End of the first part

1. Important concept

Frequency/negative frequency
Amplitude
Phase
Dirac

2. Important properties

All signals have a Fourier representation
The spectrum of a real signal is even
A periodic signal has a discrete fourier transform
A non-periodic signal (or finite support) has a continuous fourier transform

3. Attention !!!!

Fourier transform (complex valued) =/= Spectrum of a signal (real and positive)

II – Discrete signal

Convolution

The convolution operation noted $*$ between two signals s and k is:

$$(s * k)_\tau = \int_{-\infty}^{\infty} s(t)k(\tau - t)\,dt$$

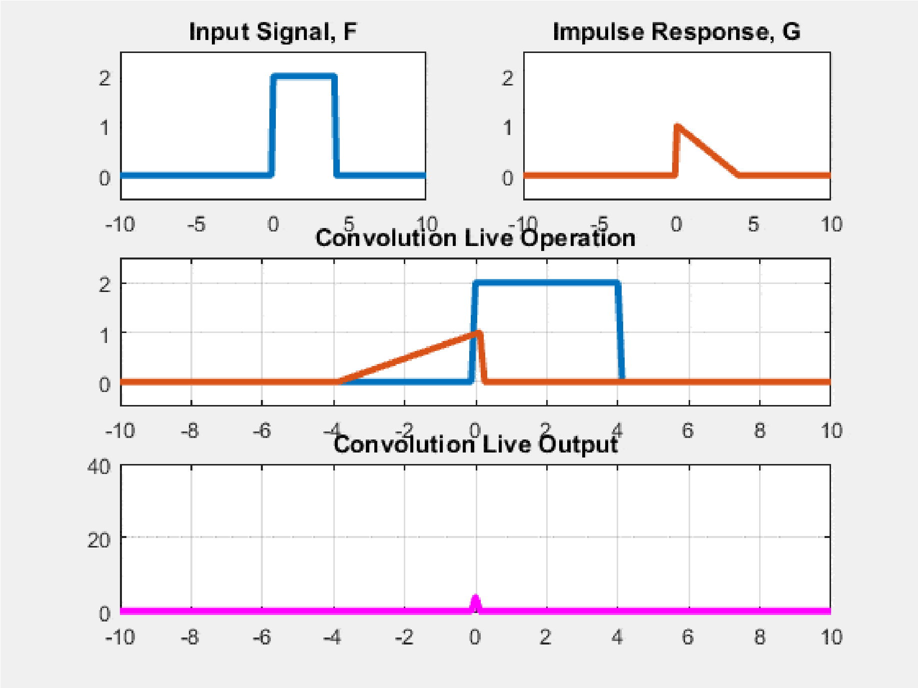
The convolution has the following fundamental property:

$$TF[sk] = TF[s] * TF[k]$$

$$TF[s * k] = TF[s]TF[k]$$

Example

$$(s * k)_\tau = \int_{-\infty}^{\infty} s(t)k(\tau - t)\,dt$$



Convolution

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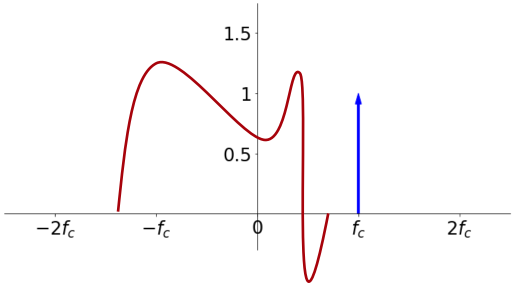
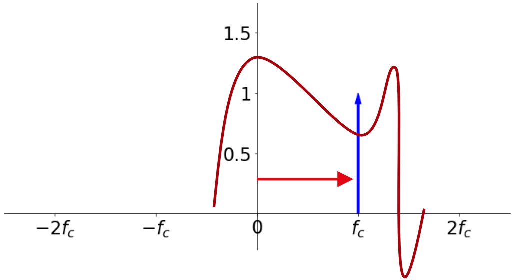
$$TF[sk] = TF[s] * TF[k]$$

$$TF[s * k] = TF[s]TF[k]$$

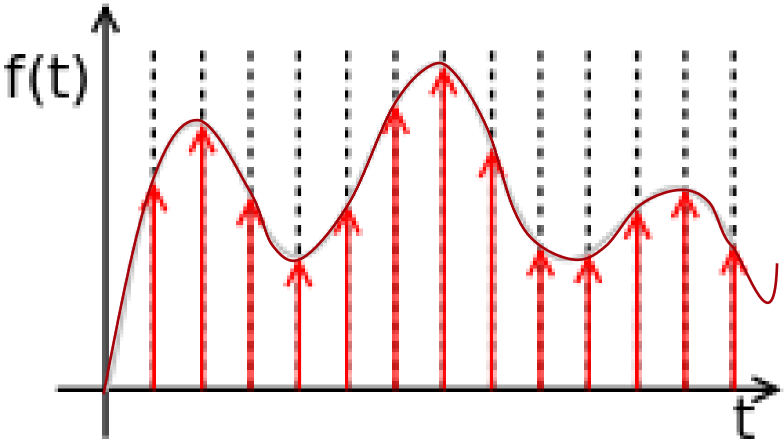
For this part we will only specify the convolution of a function with a dirac:

$$\tilde{s} = (s * \delta_T)$$
$$\tilde{s}(t) = s(t\text{-}T)$$

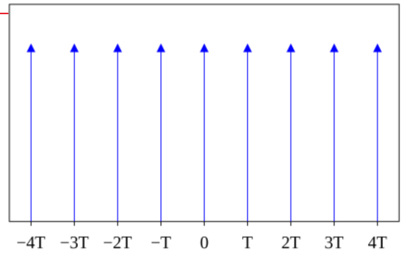




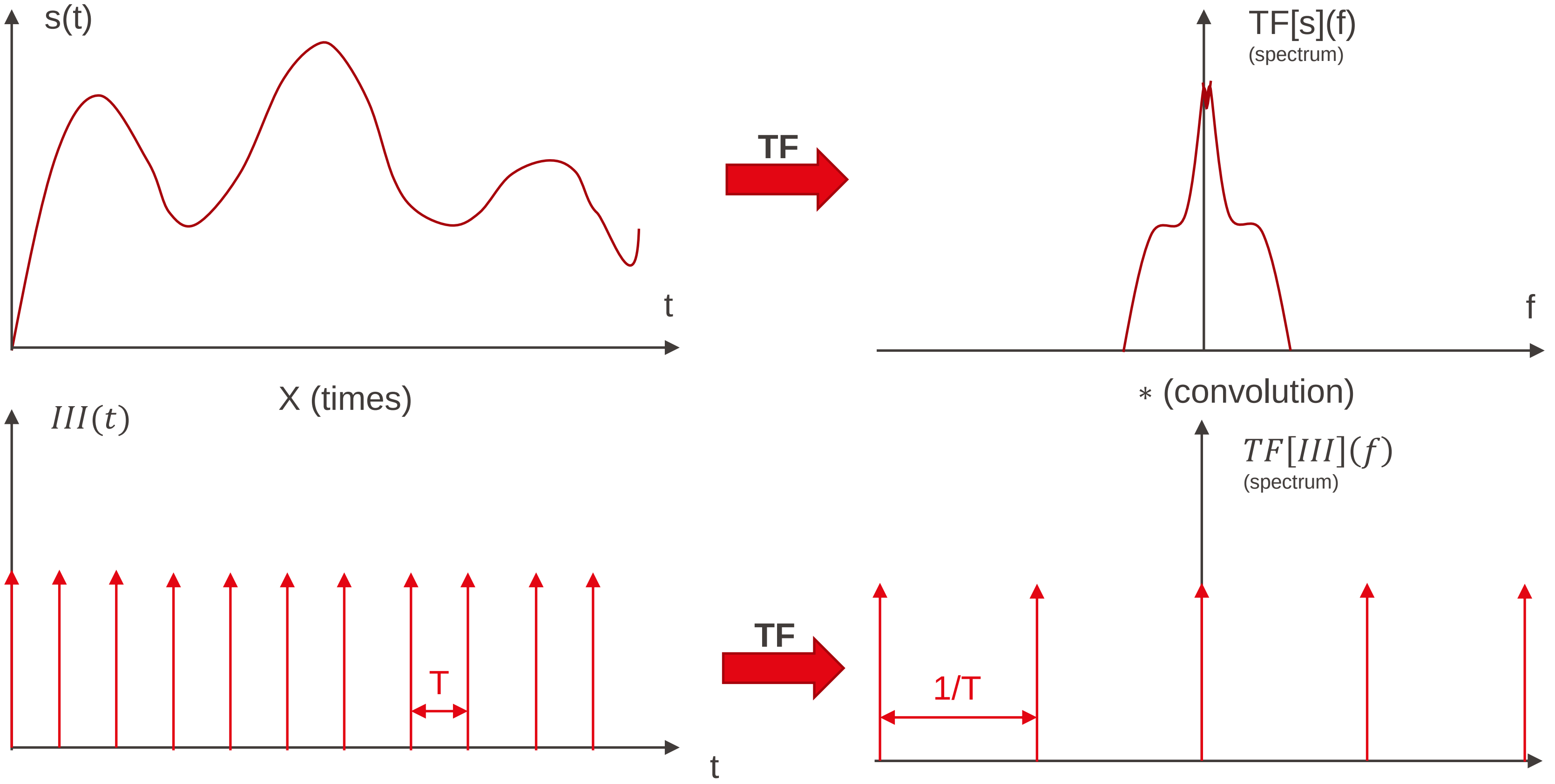
What is a discrete signal



Multiplication of a continuous signal with a Dirac comb



What is a discrete signal



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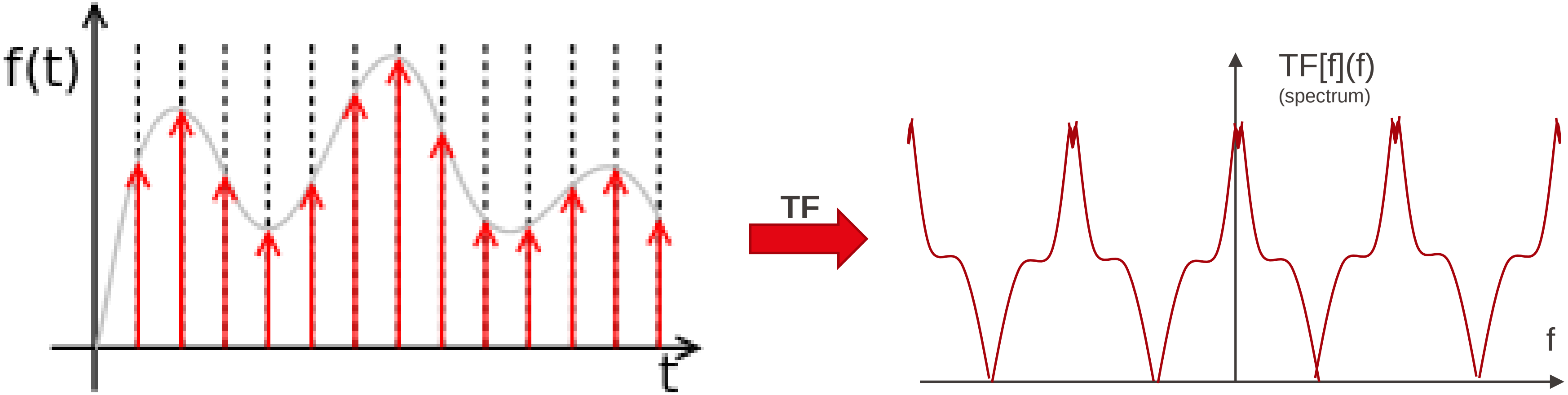
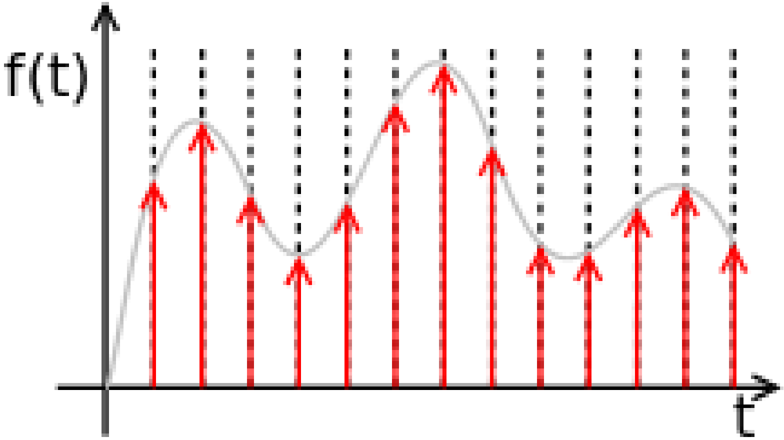
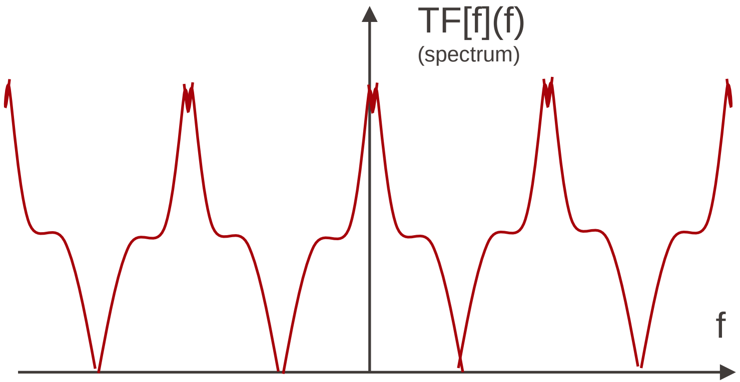
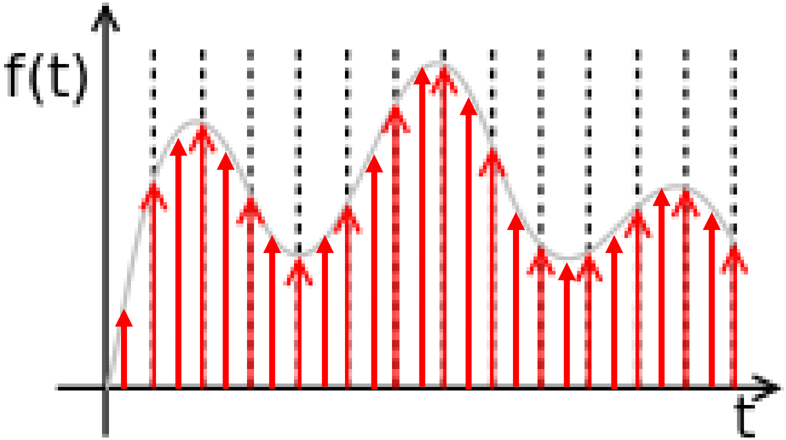


What is a discrete signal



TF

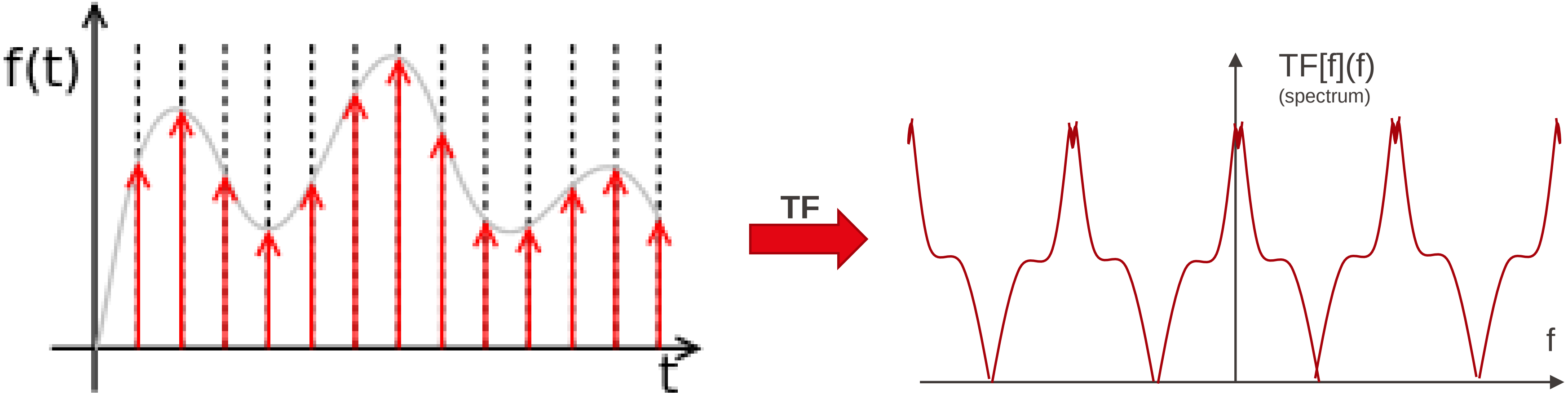


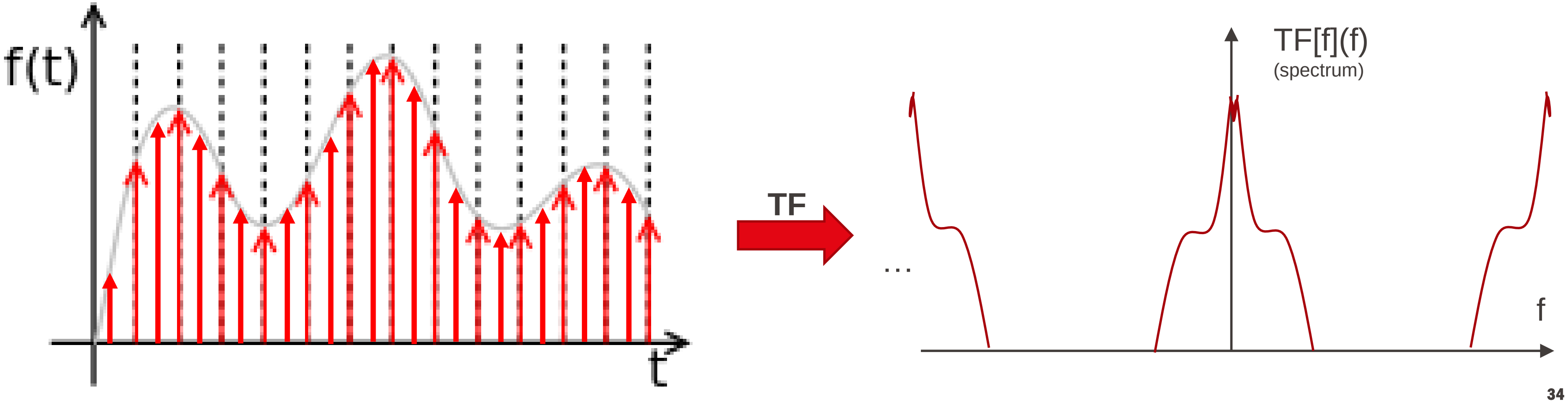


TF

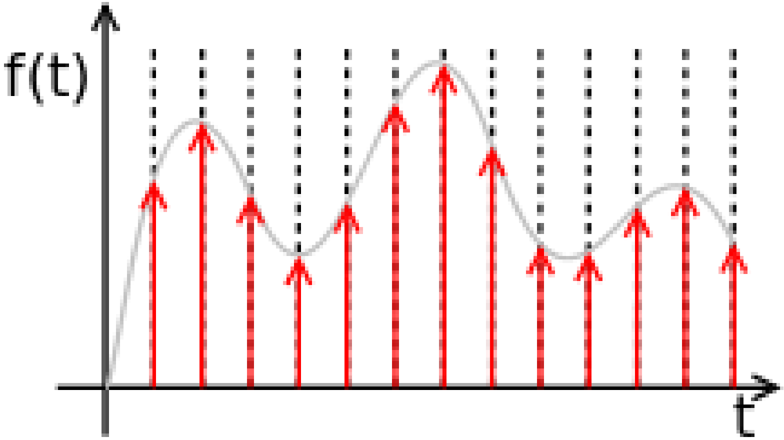
…

What is a discrete signal

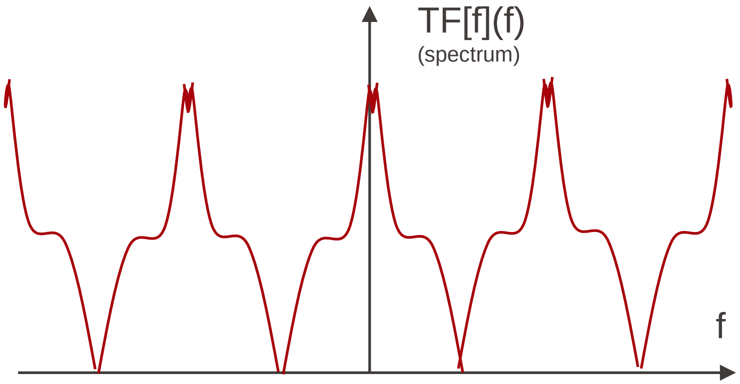


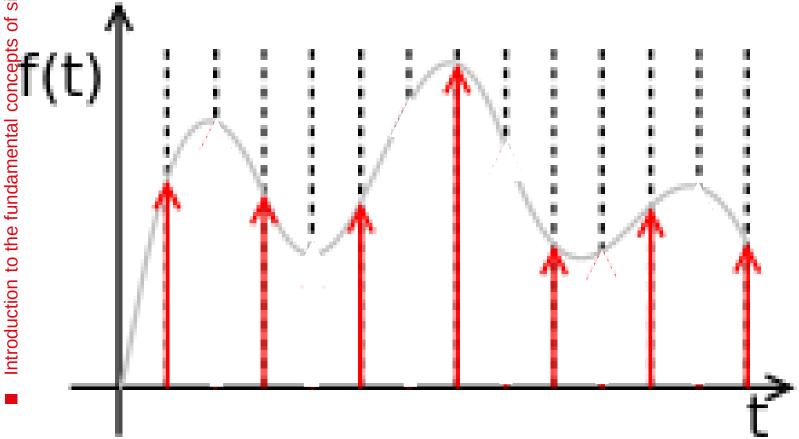


What is a discrete signal



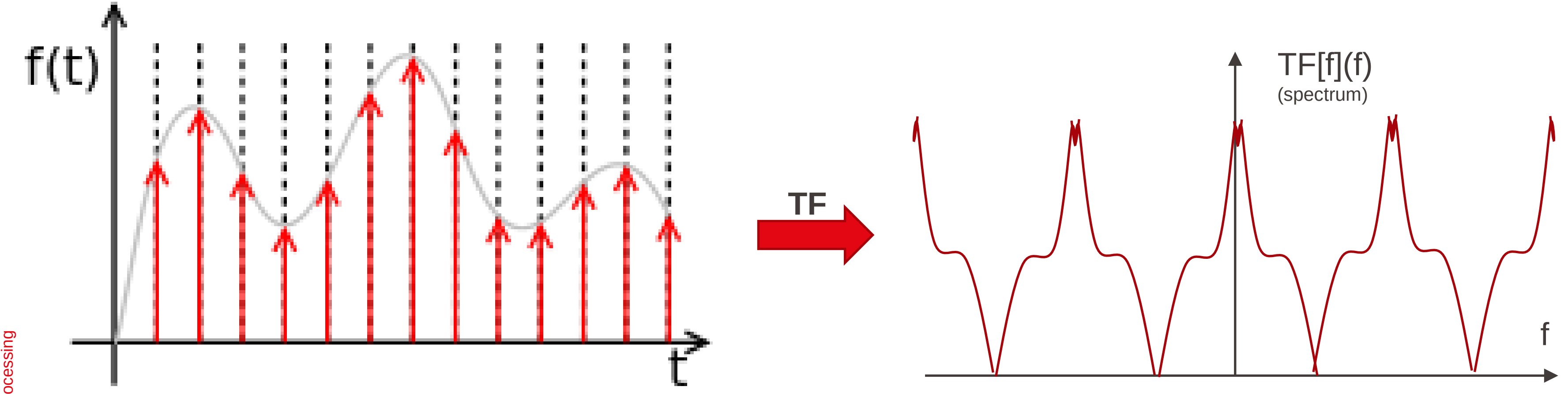
TF

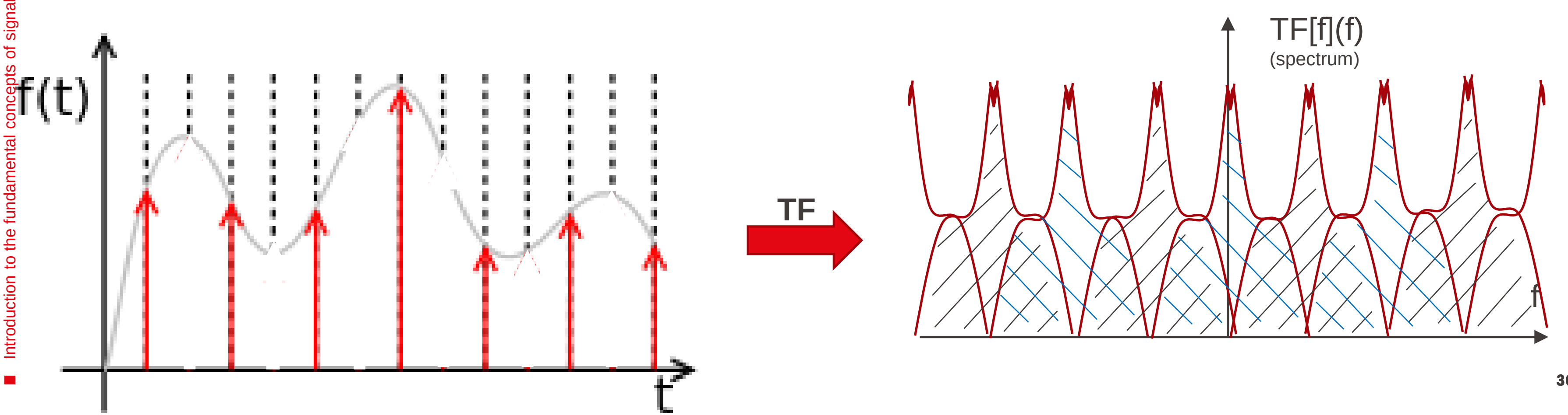




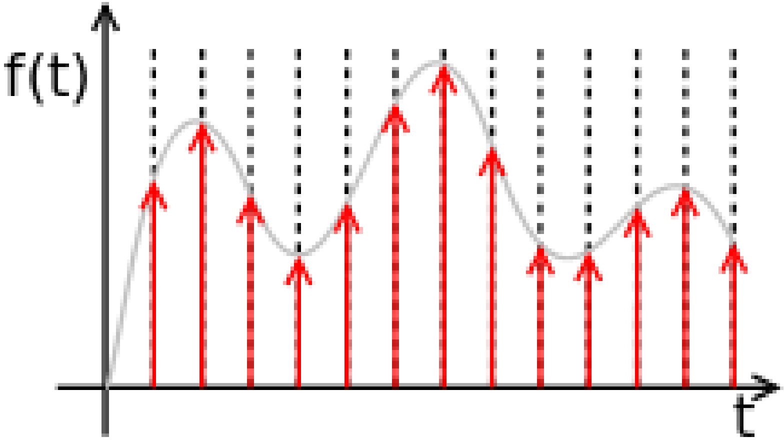
TF

What is a discrete signal

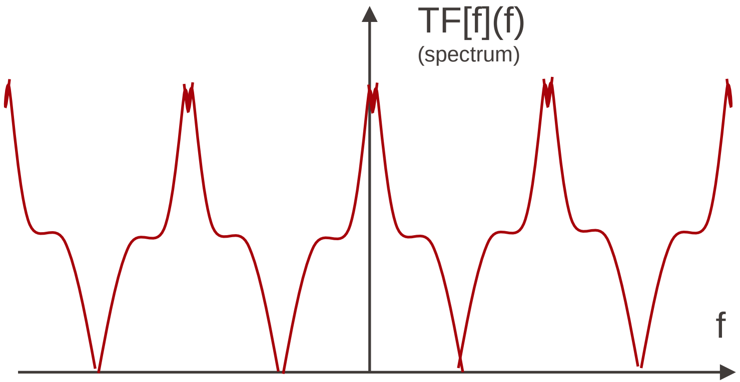


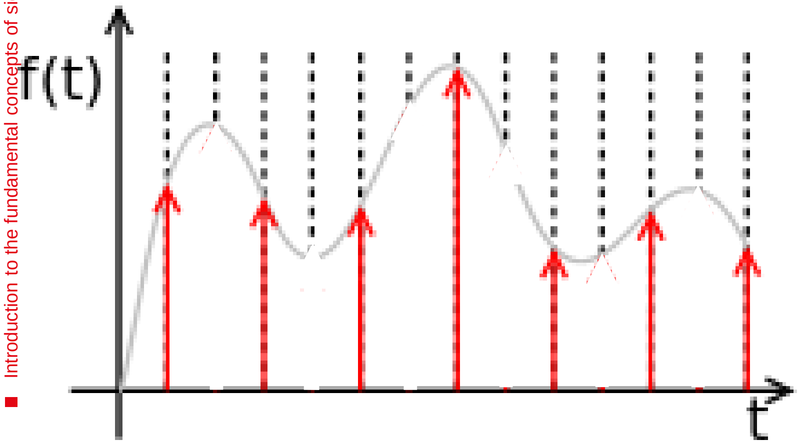


What is a discrete signal

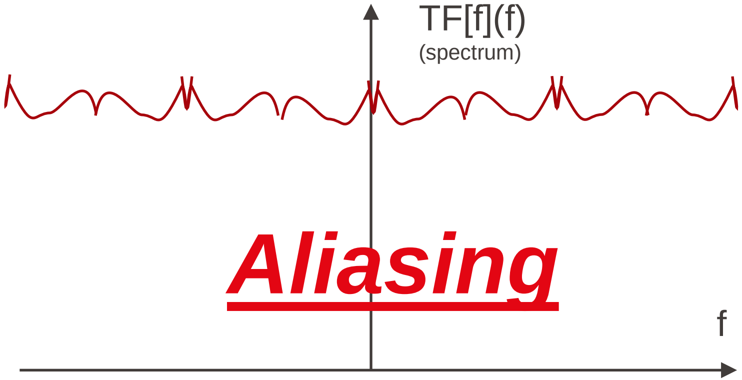


TF





TF



Moiré pattern

Apply low pass before decimation

High frequencies became low frequencies artifacts due to subsampling

Shannon Nyquist theorem

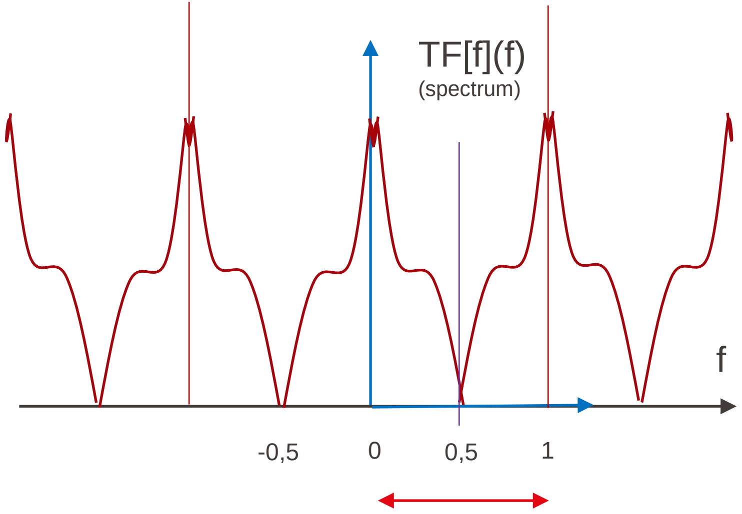
If s(t) contains no frequency higher than F_{max} Hz,

Then,

It is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2F_{max}}$

(Or the sampling frequency should be higher than $2F_{max}$)

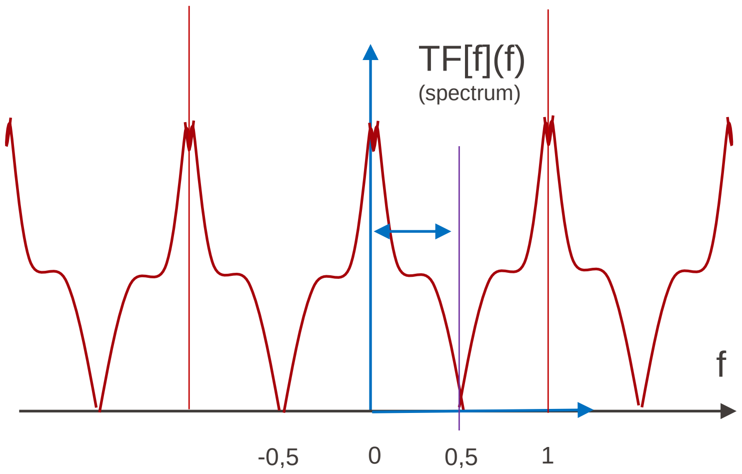
Discrete FFT representation



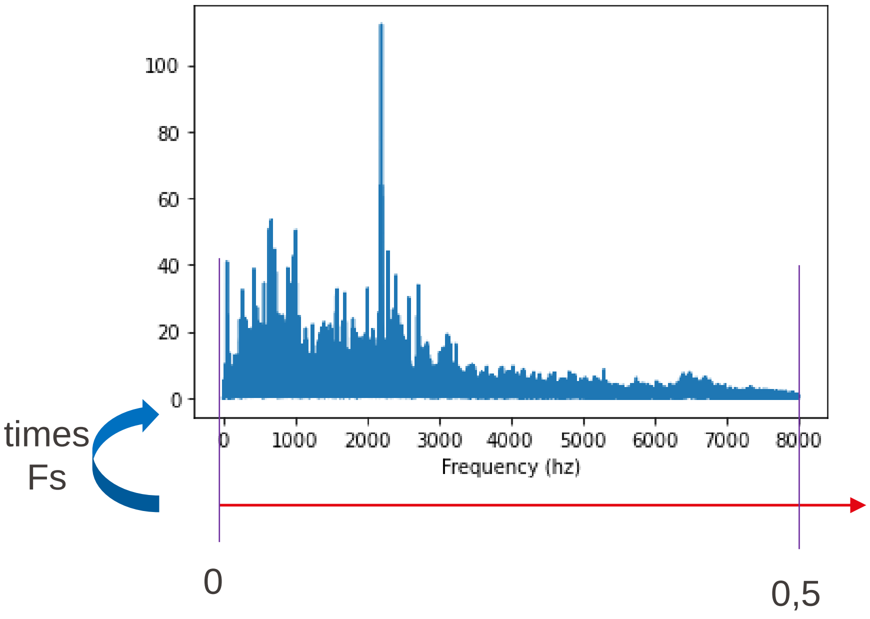
1/Ts =Fs

Max freq 0,5 Fs

Discrete FFT representation



Module of the fourier transform



End of the second part

1. Important concept

Convolution
Convolution with a dirac function
From continuous to discrete signal using a Dirac comb
Aliasing when not enough Dirac in our comb
Normalized frequencies

2. Important properties

Shannon Nyquist theorem
Convolution in time domain is multiplication in the frequency domain and vice versa

3. Attention !!!!

When applying a subsampling operation: decimate (low pass then subsample)

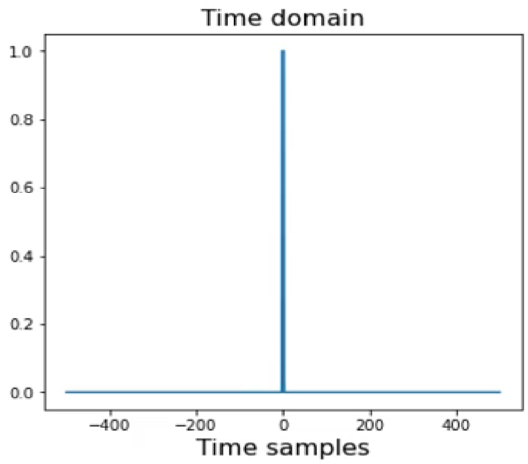
III – Time-frequency

Uncertainty principle

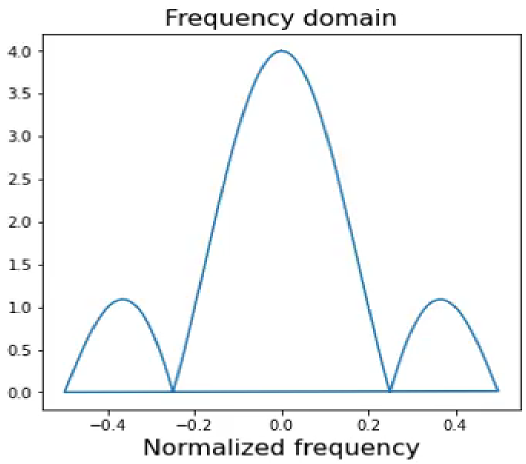
We cant be infinitely precise in time and in frequency

$$\boxed{\Delta t \cdot \Delta \omega \geq \frac{1}{2}}$$

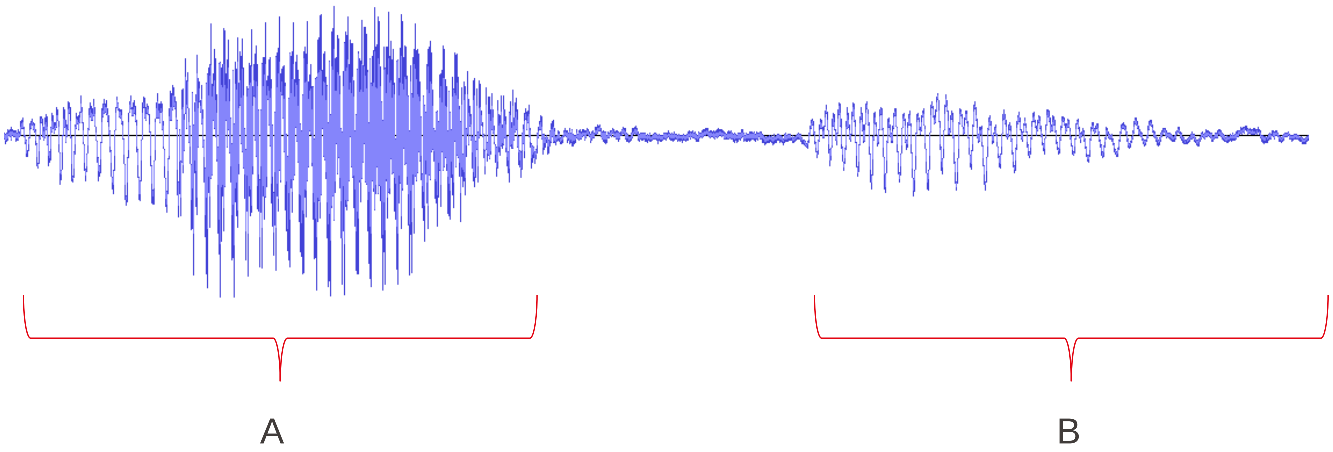
Rectangular function



Module of sinus cardinal function

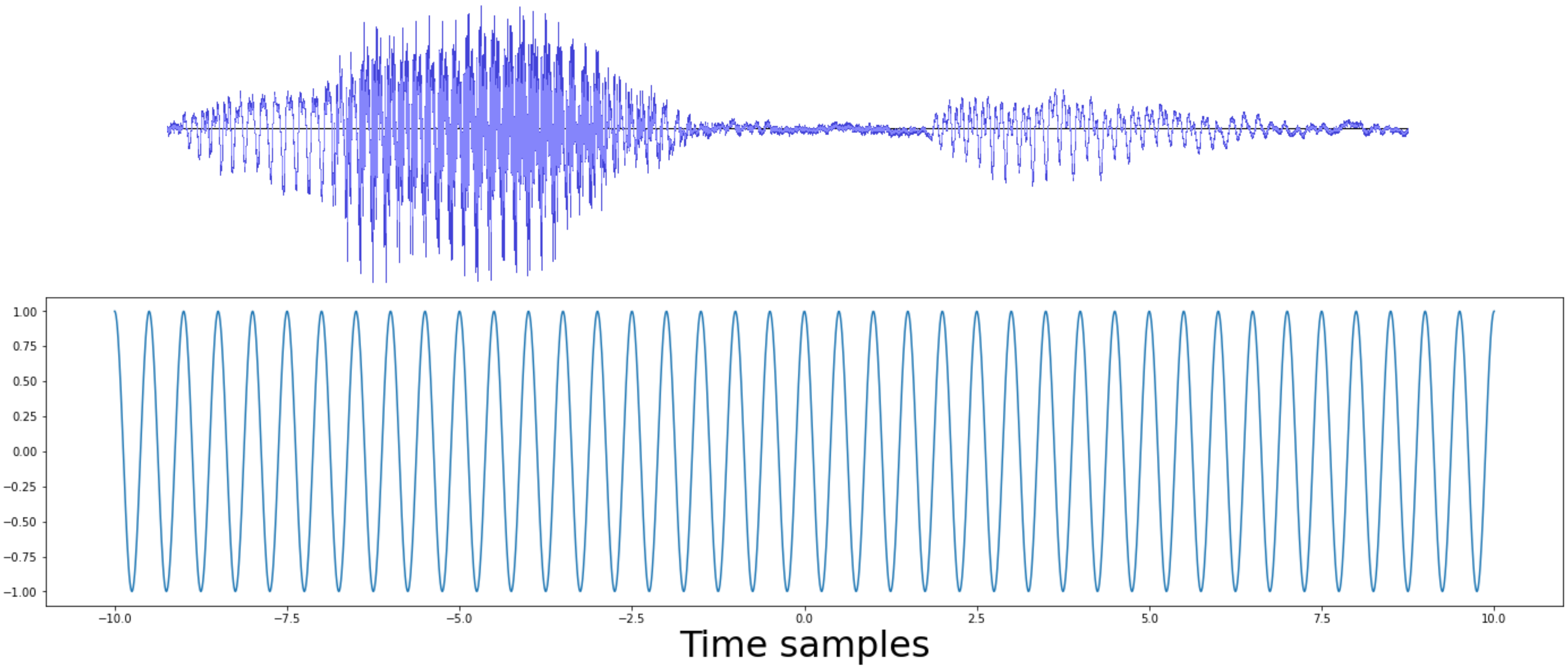


Time information loss



Applying the fourier transform on the whole signal will merge informations from part A and part B

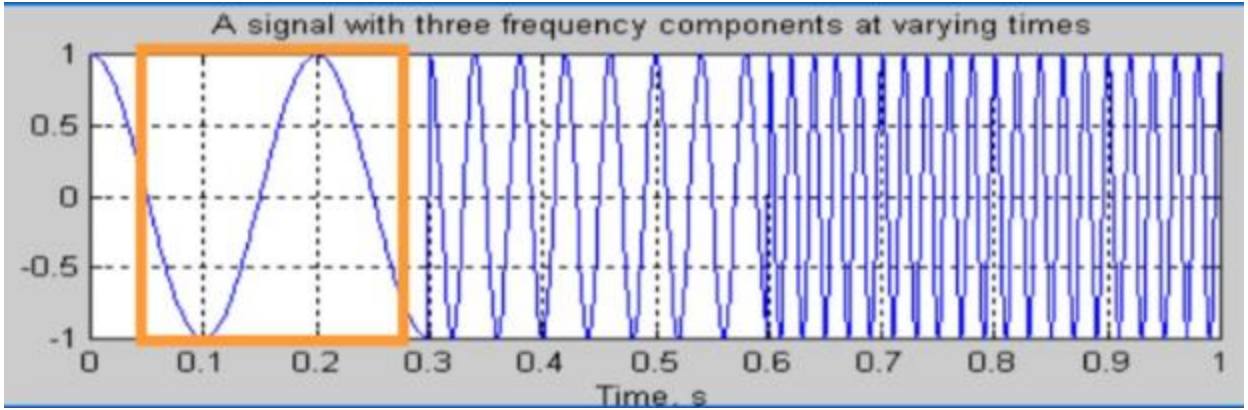
Time information loss



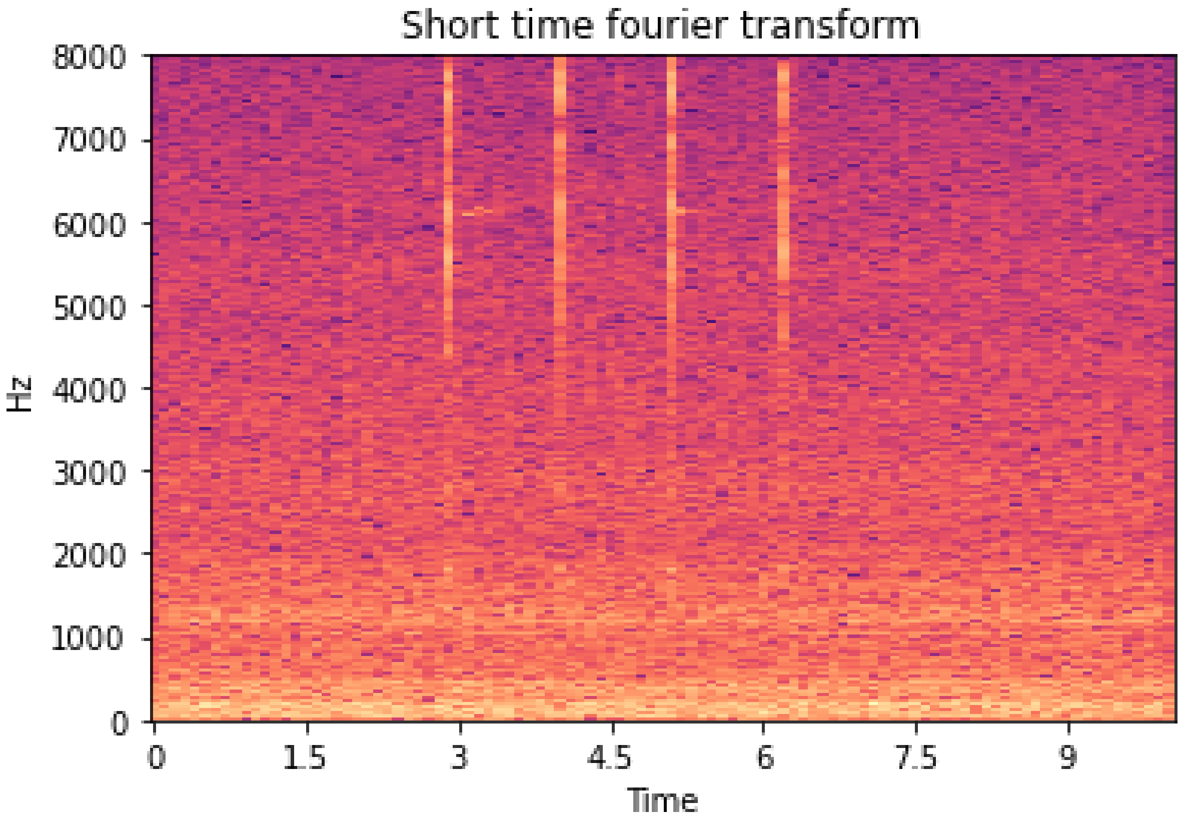
Short Time Fourier Transform

Steps

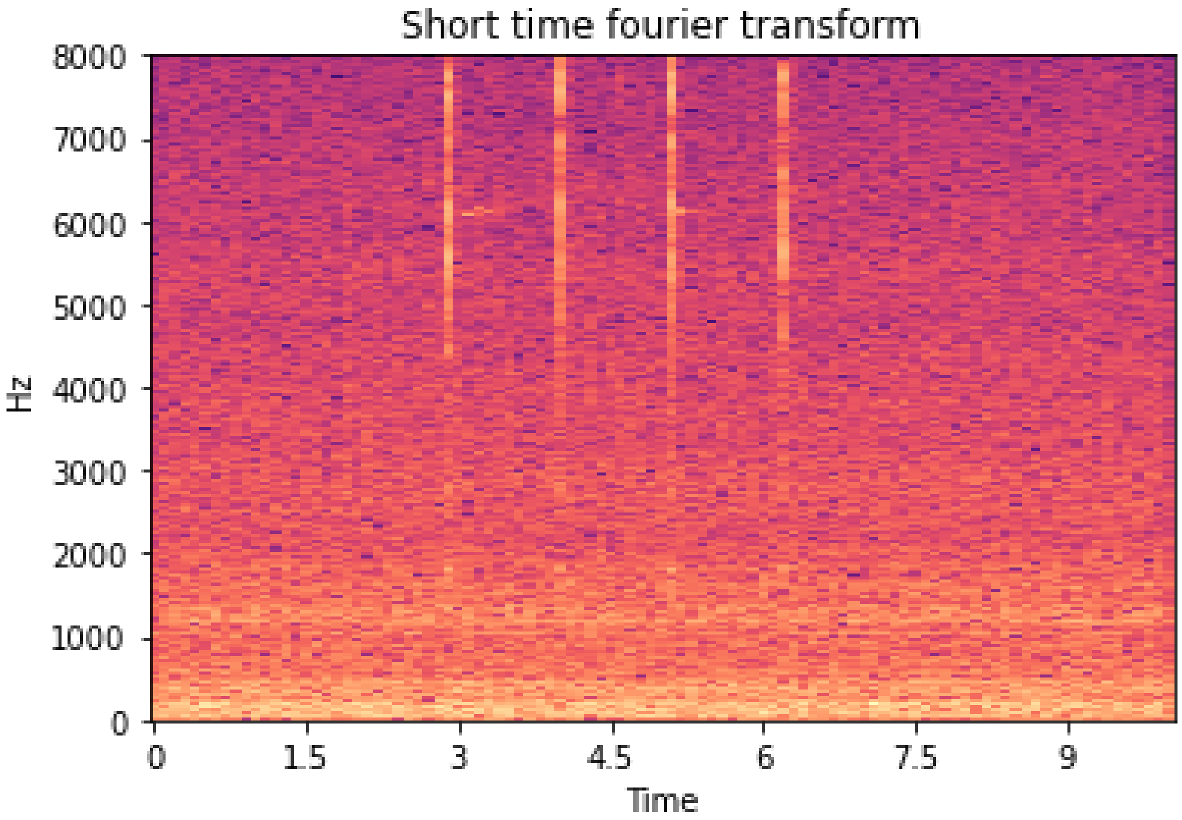
- Choose a window of finite length
- Place the window on top of signal at t=0
- Compute fourier transform of truncated signal
- Incrementally slide the windows
- Repeat until window reaches the end of signal



Spectrogram representation



Spectrogram representation

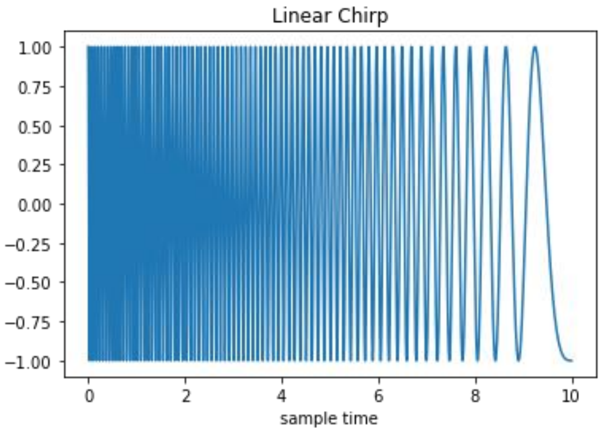


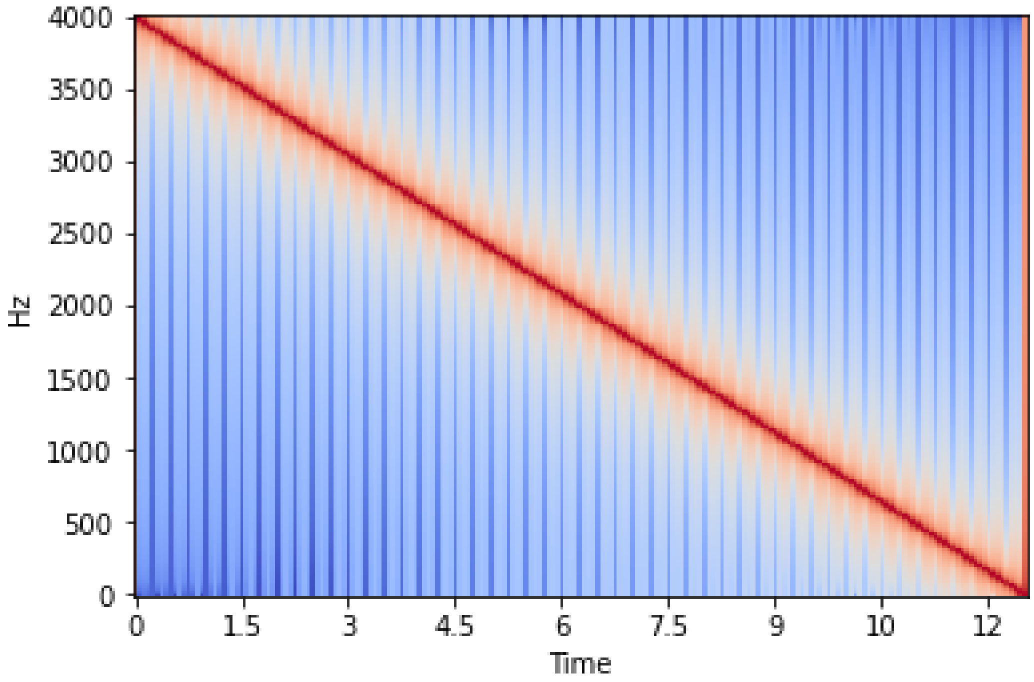
Three main parameters:

- Window size (n_fft)
- Stride (hop_length)
- Window kind

Frequency aggregation

Advantage: Reduce both time and frequency dimensions
Cons: informations are lost, impossible to reconstruct the original signal





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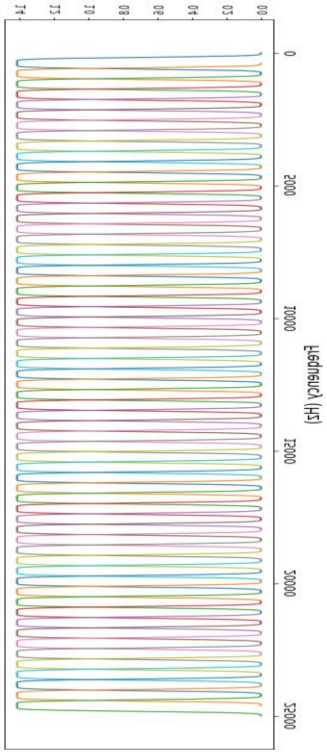
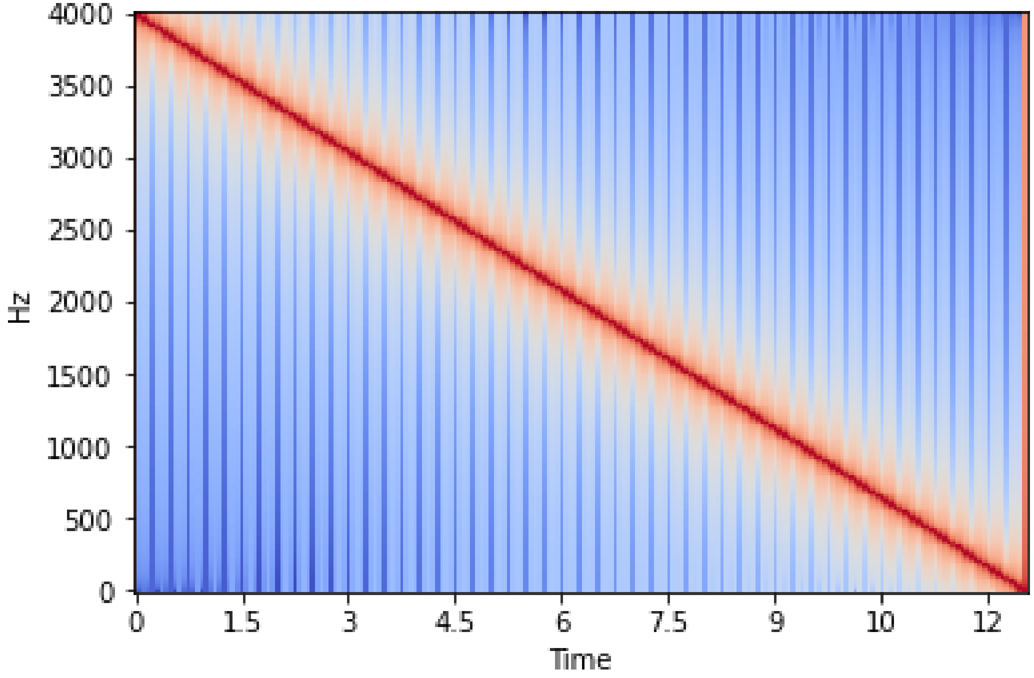
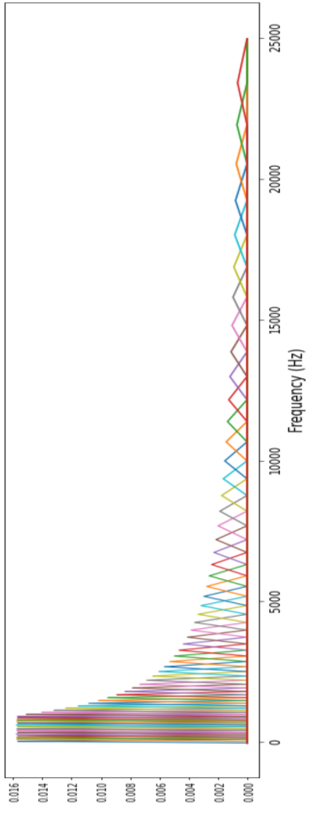
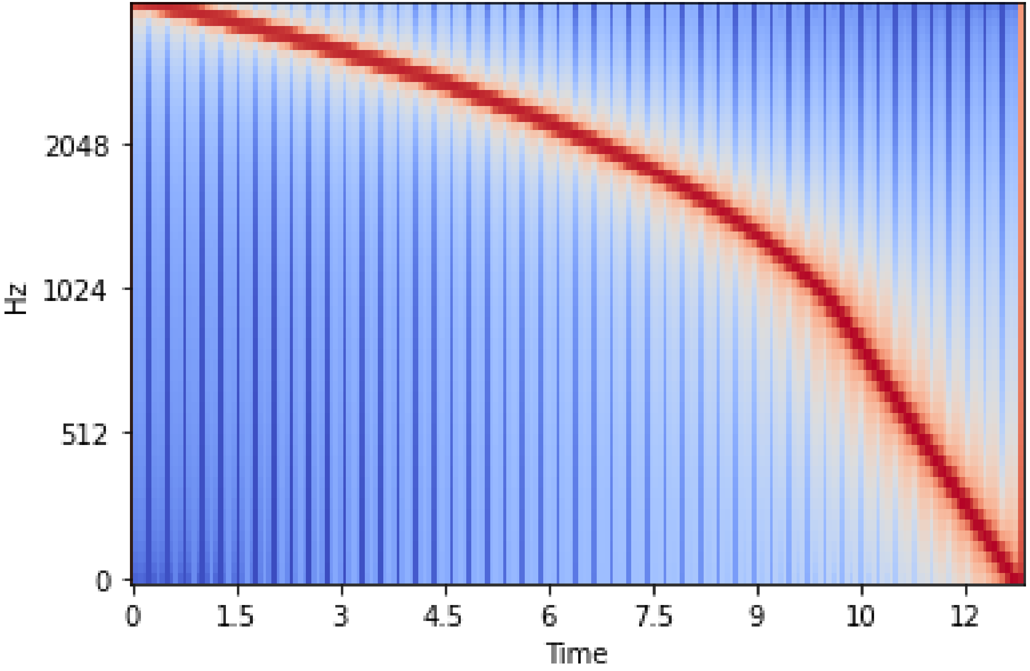




Goal: reduce the frequency dimension

Melspectrogram representation





Goal: reduce the frequency dimension, focuse more on the lowest frequencies (which often contains more usefull information)

End of the second part

1. Important concept

Time-frequency representation
Short time fourier transform
Melspectrogram

2. Important properties

Uncertainty principles