# Week #6 - Axisymmetric Elasticity: Boussinesq problem

The topic of this week's exercice is the elastic resolution of displacement and stresses beneath an uniformly loaded circular foundation (radius $R_f$). This problem has a known analytical solution, derived by Boussinesq, directly below the center of the foundation and is tabulated with charts anywhere else. We concentrate on the undrained response and solve the problem of from the initial state of the medium at rest (before the applicaiton of the load). For this exercice, we solely concentrate on the resolution of the elastic system, which after finite element discretization reduce to:

$$\mathbb{K}\mathbf{u} = \mathbf{f}$$

where $\mathbf{f}$ is the vector of total forces (i.e nodal, surface and body forces, with the later two lumped into the first), $\mathbb{K}$ the stiffness matrix and $\mathbf{u}$ the displacement vector.

Consider the problem shown in figure 1. In the first part of the file *boussinesq_plate_ axisymmetric.ipynb* we create a mesh. Additionally, a so-called "mesh fan" is introduced using the $\tanh(x)$ function. The aim is to increase the number of elements in a circular structure around the end point of the foundation (to resolve better the singularity there). Try to play around with the different parameters of the mesh refinement function to understand its precise mechanism.

## 1 Calculation of the surface traction

In the second part, we need to introduce the boundary conditions which can either be on displacement or force. The displacement boundary conditions are such that we restrict movement on the lower and the right boundary (far-field), as well as the horizontal displacement on the axis of symmetry (left boundary). Note that due to the axisymmetric problem the displacement boundary condition on the axis of symmetry is automatically verified (derive by your own why). The surface force, however, cannot simply be distributed onto the nodes but we need to lump it correctly: we need to perform a line integral on the edge where the force is applied. This line integral naturally depends on the order of the shape functions of the element. We performed this by adding linear elements to the setup and performed a numerical Gauss integration on the edge.

- Set the boundary conditions properly and perform the force check before going to the second step.

## 2 System assembly and solving

As this specific configuration fits to axissymmetric conditions we need to adapt the assembly of the stiffness matrix. Recall that the weak form of the stiffnes matrix is given by the integral:

$$\int_{\Omega^e} \mathbf{B}^T\mathbf{D}\mathbf{B}\ \mathrm{d}V = \mathbf{K}^e \tag{1}$$

where additionally we need to account for the axissymmetric case. This means that we need to transform the volume integral into a 2D integral accounting for the axisymmetry. Equation 1 changes thus to:

$$2\pi \int_{\Omega^e} \mathbf{B}^T\mathbf{D}\mathbf{B} r \ \ \mathrm{d}r\ \mathrm{d}z = \mathbf{K} \tag{2}$$

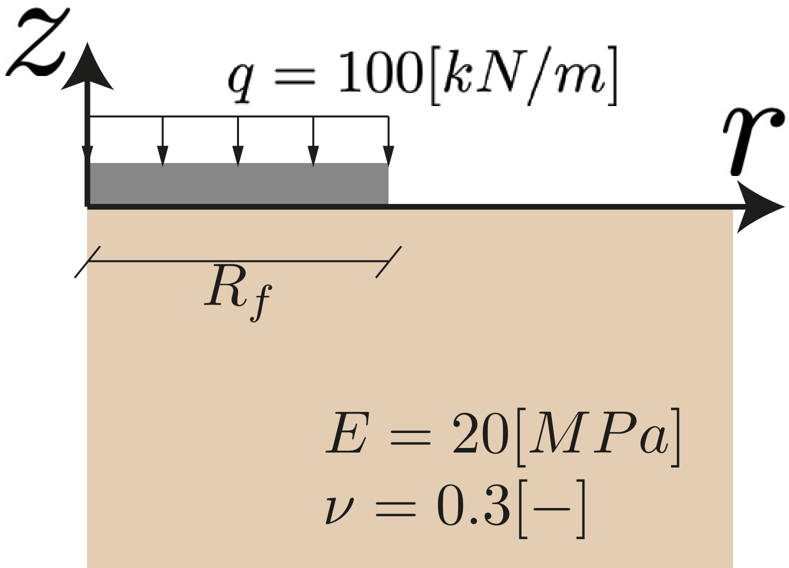


Figure 1: Sketch of the radial cut of the circular foundation. With soil properties and linear force (weight of the foundation included).

We already coded up the assembly of the stiffness matrix but encourage you to study the procedure. Remember that we use Gauss-integration upon the discretized domain and thus need to include the radius of the corresponding gauss point.

- In the next step, we will let you code up the solution of the system. Remenber to transform non-zero displacements into nodal forces. Even though we're not having such within this example code it up for the generality of the problem solution.

Next you can compare your numerical results to the theoretical solutions. The comparison is performed for a point load $P$ solution derived by Boussinesq in 1878. These analytical solutions used as approximation are given by the following formulas:

$$u_r = P\frac{1+\nu}{2\pi E}\frac{1}{R}\left(r^2|z|R^3 - (1-2\nu) * (1 - \frac{|z|}{R})\right) \tag{3}$$

$$u_z = P\frac{1+\nu}{2\pi E}\frac{1}{R}\left(2(1-\nu) + \frac{z^2}{R^2}\right) \tag{4}$$

where $R = \sqrt{r^2 + z^2}$ is the distance to the point load. This approximation is only a far field solution and will feel the boundary effects of our finite domain. Note that you must adapt $P$ as we simulate the case of a finite footing.

- We will let you extract the corresponding DOF's for the plots as to familiaries you with the set up of the system.

Then also perform a comparisons with the solution of the displacement for the case of a circular footing (of radius $R_f$) with an uniform pressure (which corresponds to the actual case simulated here). The displacement at two points $r = 0,\ R_f$ for $z = 0$ are given by

$$u_z(r=0, z=0) = q \times R_f \frac{2(1-\nu^2)}{E}$$

$$u_z(r=R_f, z=0) = q \times R_f \frac{4(1-\nu^2)}{\pi E}$$

Compute the relative error of the numerical results for this two points.

# 3 Stress calculations

We would like to do the projection of the stresses from the gauss points to the nodes as this is the most precise way of doing it. This is basically the same procedure as you've done in session 1.

- We've coded up the projection on the element level but will let you perform the assembly of the system. We have positioned the head of the function *project_stress()* within the file *MatrixAssembl.py*.

As a verification we will compare this results to the analytical solution along $z$ for $r = 0$ (Boussinesq, 1878). Note that there exist charts to compare the results at other locations as well which you can find normally in any soil mechanics textbook you like (see for example Fig. 12.9 of the TGC 18). The analytical solution for the vertical stress (which is the one we are mostly interested in) is given as:

$$\sigma_{zz} = q\left(1 - \left(\frac{1}{1 + \left(\frac{R_f}{z}\right)^2}\right)^{3/2}\right) \tag{5}$$

where $R_f$ is the radius of the foundation. Finally, we ask you to do the simulation with either linear and quadratic triangular elements. We'll let you observe how the change of the interpolation order (as well as the mesh refinement) influences your solution. If you like, try even to solve the system without the mesh fan and observe what happens.