

Exercise 9: Bifurcation in the van der Pol oscillator

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Course: Dynamical systems in biology (BIO 341)

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Consider the following two-dimensional model (the van der Pol oscillator):

$$dx/dt = y$$

$$dy/dt = -x + \mu (1 - x^2) y$$

where μ is a real constant (positive or negative). This question explores some of the conditions for there to be a bifurcation at $\mu = 0$, in which a stable spiral changes to an unstable spiral and a limit cycle appears as the parameter μ is varied (but we are not looking at the limit cycle in this problem.)

- 1) The origin (0,0) is a fixed point for any value of μ . Find the Jacobian at the origin, and evaluate its trace and determinant in terms of μ .
- 2) State why that the origin cannot be a saddlepoint for any value of μ .
- 3) On the Tau-Delta plot, mark the location of the fixed point for the following values of $\mu = -3, -2, -1, 0, 1, 3$. Label each point with the type/stability of the fixed point. For the case $\mu = -2$, find the eigenvalue(s) and eigenvector(s) and determine if the fixed point is a star or degenerate node.
- 4) For the case $\mu = -1$, draw the phase portrait including only the nullclines (with the direction of the other vector field dx/dt and dy/dt along them) and some typical trajectories (don't spend too long on the trajectories!). Can you infer the type of the fixed point from the phase portrait?
- 5) How does the fixed point change character as μ increases from negative to positive values?
- 6) Now consider the similar equations:

$$dx/dt = y + \mu x$$

$$dy/dt = -x + \mu (1 - x^2) y$$

6.1) Find the Jacobian for the fixed point at the origin, and the fixed point's type and stability for $\mu = -3, -1, 0, 1, 3$. What can you say about the type of the fixed point at the origin by examining τ , Δ and $\tau^2 - 4\Delta$?

6.2) Are there other fixed points than the origin for this model? If so, find their coordinates (x^*, y^*) as a function of μ . Set $\mu = 0.1$ and find the coordinates of any non-zero fixed points.

6.3) For the value $\mu = 0.1$, draw the phase portrait including: the nullclines, fixed point(s), and direction of the vector field on the nullclines (no trajectories).