

BIO 341 Graded exercise 2 (30 points)

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All working by hand and calculations need to be shown.

Oscillating chemical reactions

Consider the following two-dimensional model that represents the oscillations in the concentrations of two chemical species:

$$dx/dt = I - (b + I) x + a x^2 y$$

$$dy/dt = b x - a x^2 y$$

where $x = [X]$, and $y = [Y]$ are the concentrations of the species (≥ 0), and $a, b > 0$ are constants. This is an example of an autocatalytic chemical reaction (x promotes its own production via the x^2y term) which are common in living systems, but rare in nonliving ones. They seem to be crucial for making the biochemistry of life robust.

(Note that this exercise is longer than one that could appear on the final exam, but the parts of the question are typical of exam questions. Also, because this is a take-home exercise, you could use python or any software to find the phase portrait and the trajectories, but they must be **drawn by hand** on the submitted plot. In the final exam, everything must be done by hand.)

- 1) Identify the types of the terms in the equations, (i.e., constant production/decay, linear growth/decay, competition, anti-competition, saturated competition, etc.) (2.5 points)
- 2) Find equations for the x, y null clines, and equations for the direction of the vector field along the nullclines. (4 points)
- 3.1) Find the coordinates (x^*, y^*) of the fixed point in terms of the parameters a, b and accurately indicate its location on the phase portrait. (1.5 points)
- 3.2) Draw the nullclines and the direction of the vector field along the nullclines. (3 points)
- 4.1) Find the Jacobian of the model in terms of a, b, x, y . (4 points)

4.2) Evaluate the Jacobian at the fixed point from part 3, and give its trace and determinant. (3 points)

4.3) Define the *critical value* of b to be that value (b_c) that makes the trace of the Jacobian at the fixed point zero for a given value of a , and find an expression for b_c in terms of a . (1 point)

5) Find an expression for the eigenvalues of the Jacobian at the fixed point in terms of a , b , and evaluate them for $b = b_c$. What is the type of the fixed point precisely at $b = b_c$ (3 points)

6) Draw the phase portrait **by hand** for $a = b = 1$ on a new graph (not the same one as part 2), making sure you include the following elements (8 points):

- label the nullclines
- mark the fixed point with its type/stability
- draw the vector field on all sections of the nullclines
- the direction of the vector field on both axes, near the axes, and as x, y go to infinity
- draw **representative** trajectories in all regions of the first quadrant (i.e., small x, y ; small x and large y ; large x and small y ; large x and y , etc).

8/11/24

1)

$$\dot{x} = 1 - (b+1)x + ax^2y$$

$a, b > 0$

x, y are concentrations

so also > 0

↑ source ↑ linear decay ↑ promotion
anti-competition

$$\dot{y} = bx - ax^2y$$

↑ promotion/
anti-competition ↑ competition

2)

nullclines

$$\dot{x} = 0 \Rightarrow 1 - (b+1)x + ax^2y = 0$$

$$\therefore y = \frac{(b+1)x - 1}{ax^2}$$

x nullcline

$$\dot{y} = 0 \Rightarrow bx - ax^2y = 0$$

$$\therefore x = 0 \text{ or } y = \frac{b}{ax}$$

y nullclines

What is $\dot{y}$ along $\dot{x} = 0$?

$$\dot{y} = x(b - ax(y)) \quad \text{and} \quad y = \frac{(b+1)x - 1}{ax^2}$$

$$\Rightarrow \dot{y} = x \left[b - \cancel{ax} \cdot \left(\frac{(b+1)x - 1}{\cancel{ax^2}} \right) \right]$$

$$= x \left[b - \frac{(b+1)x - 1}{x} \right]$$

$$= x \left[\cancel{b} - (b+1) + \frac{1}{x} \right]$$

$$\therefore \dot{y} = x \left(\frac{1}{x} - 1 \right) = 1 - x$$

and $\ddot{x}$ along $\dot{y} = 0$? i.e. along $x=0$ and $y = \frac{b}{ax}$

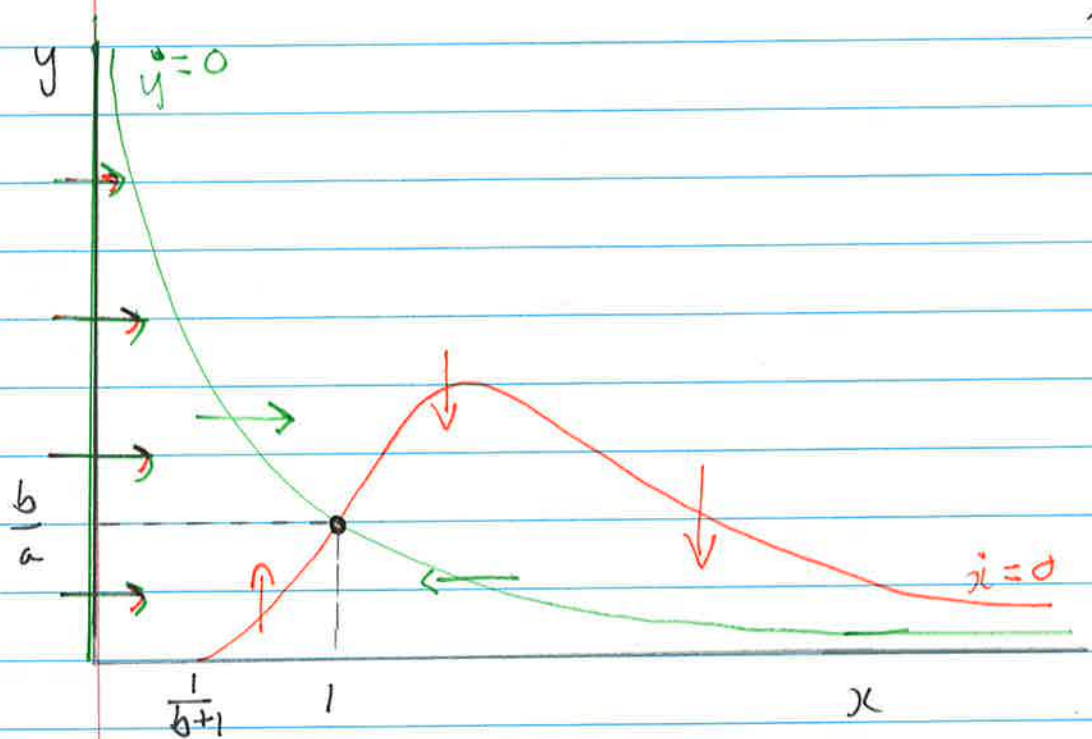
$$\Rightarrow \ddot{x} = 1 - (b+1)x + \cancel{ax^2} \cdot \frac{b}{\cancel{ax}}$$

$$= 1 - (b+1)x + bx$$

$$\therefore \ddot{x} = 1 - x$$

and for $x=0 \Rightarrow \ddot{x} = 1$

3.2)



3.1)

Fixed point (x^*, y^*)

$$\frac{(b+1)x - 1}{ax^2} = \frac{b}{ax}$$

$$\Rightarrow (b+1)x - 1 = bx$$

$$\therefore x^* = 1 \quad \Rightarrow y^* = \frac{b}{a} \quad \underline{\underline{(1, b/a)}}$$

4.1)

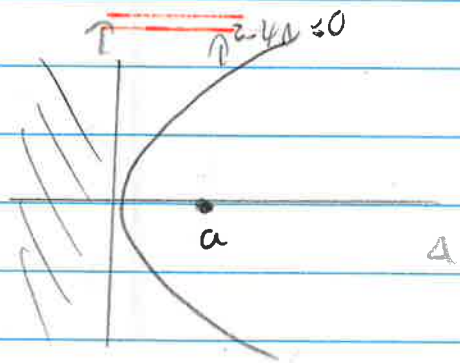
$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -(b+1) + 2axy & ax^2 \\ b - 2axy & -ax^2 \end{pmatrix}$$

4.2)

$$J|_{(1, b/a)} = \begin{pmatrix} -(b+1) + 2a \cdot \frac{b}{a} & a \\ b - 2a \cdot \frac{b}{a} & -a \end{pmatrix} = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}$$

$$\tau = b - 1 - a$$

$$\Delta = -(b-1)a + ab = a > 0$$



$$4.3) \quad \tau = 0 \Rightarrow b_c - 1 - a = 0$$

$\therefore \underline{b_c = 1 + a}$ is the critical value of b .

$$5) \quad \text{Given } \underline{J(1, b/a)} = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}$$

$$\begin{vmatrix} b-1-\lambda & a \\ -b & -a-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda + 1 - b)(\lambda + a) + ab = 0$$

$$\Rightarrow \lambda^2 + a(1-b)\lambda + (1+a-b)\lambda + ab = 0$$

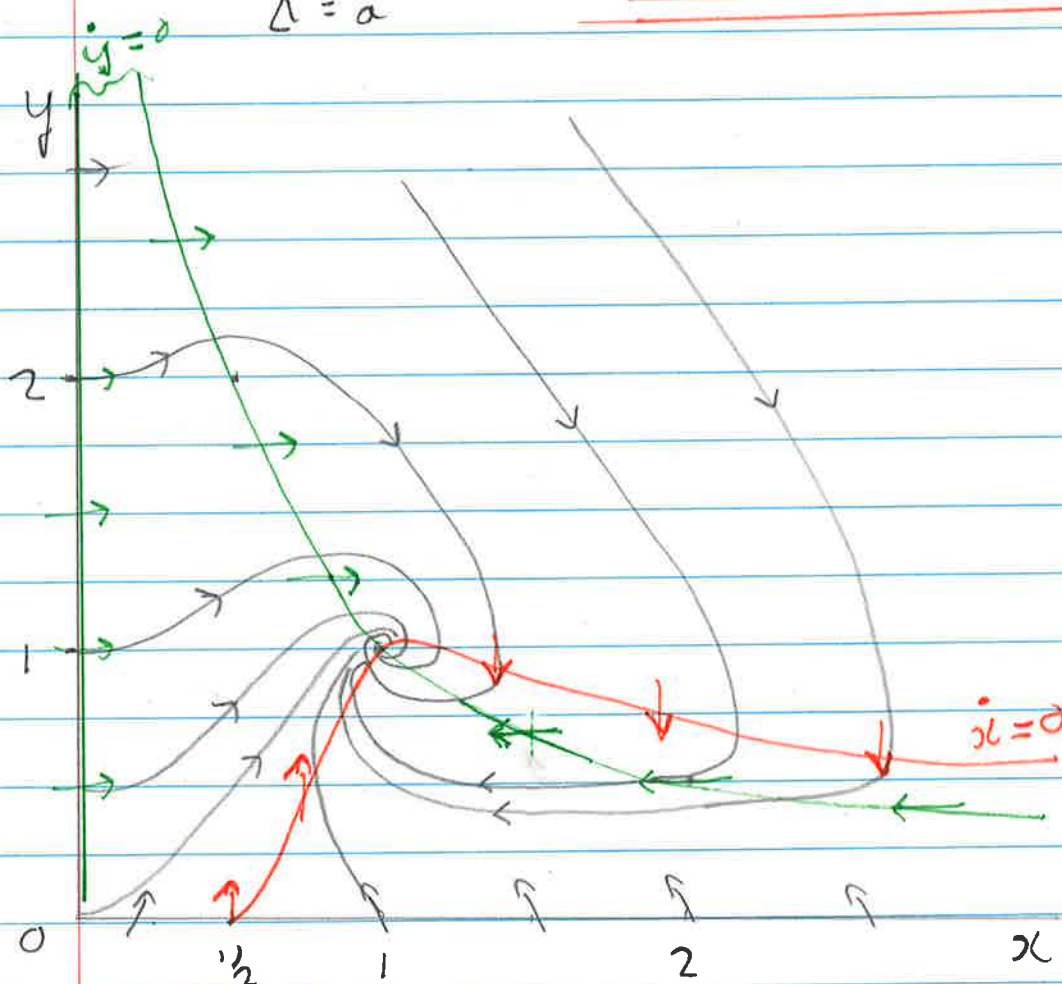
$$\therefore \lambda^2 + (1+a-b)\lambda + a = 0$$

$$5) \quad \lambda = \frac{(b-1-a) \pm \sqrt{(1+a-b)^2 - 4a}}{2}$$

$$\text{When } b = b_c = 1+a$$

$$\Rightarrow \lambda = \frac{0 \pm \sqrt{0-4a}}{2} = \pm i\sqrt{a}$$

Because $\tau = 0$ we have a centre at $(1, \frac{1+a}{a})$
 $\Delta = a$



$a=b=1 \Rightarrow \tau=-1$ $\therefore$ Stable spiral at $(1, 1)$.
 $\Delta=1$

VECTIAL FLOW DIRECTIONS

Fixed point is at $(1, 1)$.

$$\dot{x} = 1 - 2x + x^2y$$

$$\dot{y} = x - x^2y$$

At $x=0 \Rightarrow \dot{x} = 1$ all along y axis / it's a nullcline!
 $\dot{y} = 0$

$$\text{But } x = \frac{1}{2} \Rightarrow \dot{x} = 1 - 1 + \frac{1}{4}y = \frac{1}{4}y > 0 \text{ for all } y$$

$$\dot{y} = \frac{1}{4}(2 - y)$$

$$x = \frac{1}{4} \Rightarrow \dot{x} = 1 - \frac{1}{2} + \frac{1}{16}y = \frac{1}{2} + \frac{1}{16}y > 0$$

$$\dot{y} = \frac{1}{16}(4 - y) > 0 \text{ for } y < 4$$

What about $y=0$?

$$\dot{x} = 1 - 2x > 0 \text{ for } x < \frac{1}{2} \text{ otherwise } < 0$$

$$\dot{y} = x > 0 \text{ for all } x$$

What about x, y large?

$$\left. \begin{array}{l} \dot{x} \rightarrow x^2y \\ \dot{y} \rightarrow -x^2y \end{array} \right\} \frac{dy}{dx} \rightarrow -1$$

8.31

