Background quiz

Background quiz: go.epfl.ch/turningpoint

Session Id: julian23

All input is anonymous; data are stored outside CH

Break

Lecture 3 Recapitulation

$$dg/dt = s - r*g + g^2 / (1 + g^2)$$

Case s=0:

No basal production, 1-3 FPs, $g^*=0$ always one of them.

Case s>0:

Again 1-3 FPs, but $g^*=0$ is not one.

Location of FPs depends on intersection of the functions:

$f(g) = r\,g - s$

$h(g) = g^2 / (1 + g^2)$

Hysteresis in r; gene expression jumps from low g^* to high g^* and vice versa at different values of r.

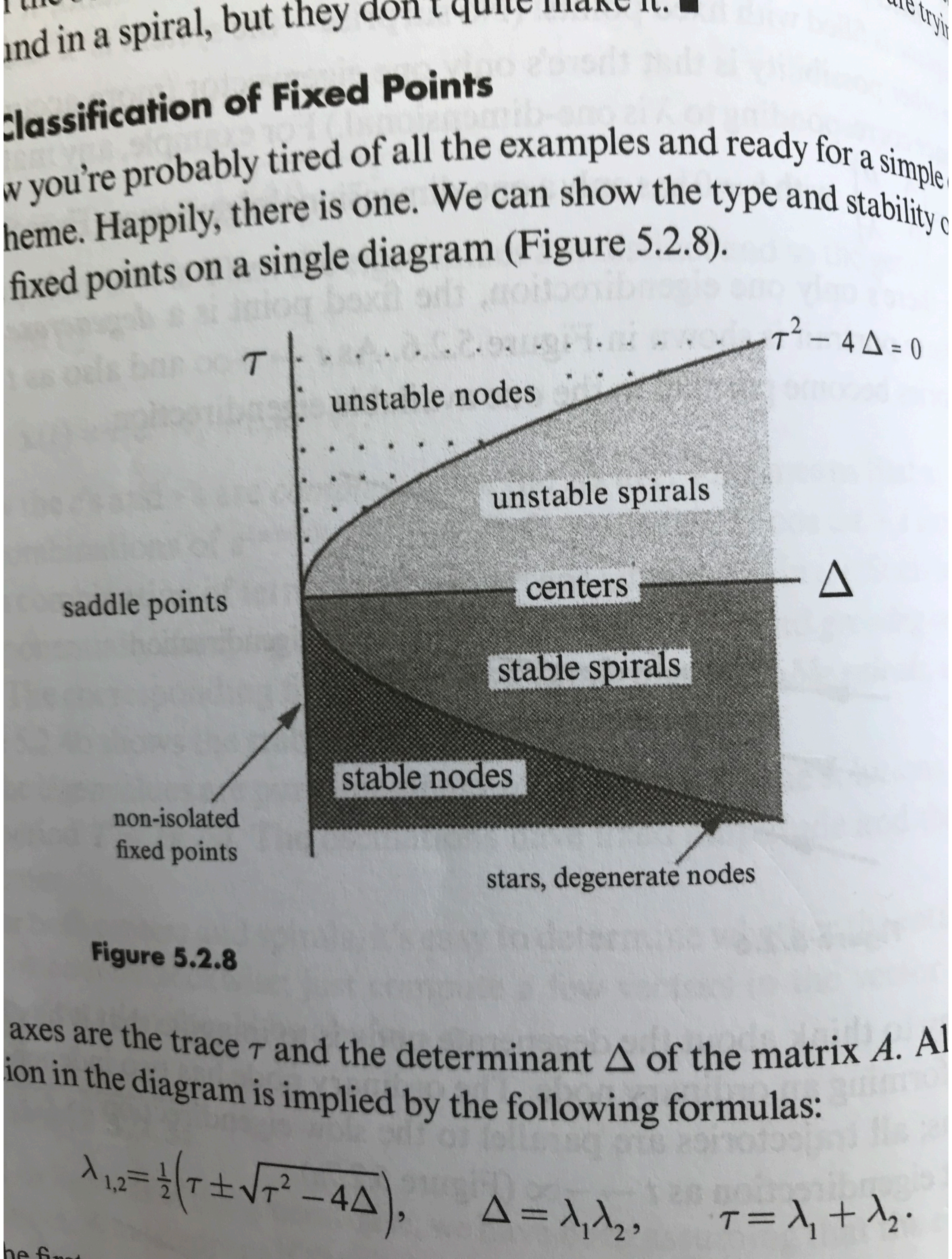
Newton's method for finding zeroes:

$x_{n+1} = x_n - f(x_n) / f'(x_n)$

Start with x0 close to the zero, and be careful if $f'(x_n) \sim 0$.

Lecture 4 Introduction

- There is only one 1D linear ODE: $dx/dt = ax$, but 2D linear problems are so varied, we need a scheme to classify them.
- 2D *non-linear* problems are reducible to a (locally) independent set of 2D *linear* problems; so solving a 2D linear problem is nearly all you need to solve 2D non-linear problems.
- Each higher dimension allows new phenomena: in 2D we get more complex fixed points than stable/unstable nodes, e.g., saddlepoints and spirals
- Visualize phase portraits using a 2D Runge-Kutta integration scheme



nd in a spiral, but they don't quite make it. ■

lassification of Fixed Points

w you're probably tired of all the examples and ready for a simple
heme. Happily, there is one. We can show the type and stability
fixed points on a single diagram (Figure 5.2.8).

Figure 5.2.8

axes are the trace τ and the determinant Δ of the matrix A. Al
ion in the diagram is implied by the following formulas:

$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1 \lambda_2, \qquad \tau = \lambda_1 + \lambda_2 .$$

Strogatz, ch. 5

Given the 2D linear dynamical system:

d**X**/dt = **M X**

where the matrix **M =** (a, b, c, d).

Calculate the trace τ and determinant Δ of M:

τ = a + c
det M = ad - bc

Draw XY axes with Δ as the X axis and τ as the Y axis.

Add the quadratic curve τ**2 - 4 Δ = 0 on rhs.

Locate the point (Δ, τ) on the graph, and read off the type and stability of the fixed point.

NB For fixed points on the curves, linear stability analysis fails for nonlinear 2D systems.

Classification of Fixed Points in 2D linear systems

Case $\Delta < 0$, saddle point, eigenvalues have opposite signs.

Case $\Delta = 0$, at least one eigenvalue is zero; non-isolated fixed points.

Case $\Delta > 0$, two sub-cases:

$\tau < 0$ stable fixed points

$\tau < 0$ and $\tau^{**}2 - 4^{*} \Delta > 0$, FP is a stable node, both eigenvalues are negative
$\tau < 0$ and $\tau^{**}2 - 4^{*} \Delta < 0$, FP is a stable spiral, complex eigenvalues, real part negative

$\tau = 0$, centres, trajectories rotate around the FP

$\tau > 0$ unstable fixed points

$\tau > 0$ and $\tau^{**}2 - 4^{*} \Delta > 0$, FP is a unstable node, both eigenvalues are positive
$\tau > 0$ and $\tau^{**}2 - 4^{*} \Delta < 0$, FP is a unstable spiral, complex eigenvalues, real part positive

If $\tau^{**}2 - 4^{*} \Delta = 0$, we have stars or degenerate nodes. If there is a single eigenvector. it’s a degenerate node, if every vector is an eigenvector, it’s a star.

Recipe for solving the general 2D linear dynamical system

1) Given d**X**/dt = **M X**, the type of FP at the origin is determined by the trace (τ) and determinant (Δ) of **M**.

2) Draw the Δ-τ plot from previous slide, and add the curve τ**2 - 4*Δ = 0 on the right hand side. Locate the fixed point of **M** by its (Δ , τ) value, and read off its type and stability.
(if you just want the type/stability of FP, stop here)

3) On the phase portrait, draw the nullclines where dx/dt = 0 or dy/dt = 0. Mark the direction of flow of the other component of the vector field on the nullclines.

4) Find the eigenvalues, λ_1 λ_2 , and eigenvectors, $\mathbf{V_1}$ $\mathbf{V_2}$, of the matrix **M**.

5) Draw the eigenvectors of **M** (they are always straight lines in the plane). and fill in the vector field around the fixed point

Trajectories approach (leave) a stable (unstable) node parallel to the slow eigendirection (smallest magnitude eigenvalue) as time goes to plus infinity, and become parallel to the fast eigenvector as time goes to minus (plus for unstable) infinity

Trajectories approach the unstable manifold of a saddlepoint as time goes to infinity, and the stable manifold as time goes to minus infinity.
(if you just want the qualitative solution, stop here)

6) Write the general solution as: $\mathbf{X}(t) = c_1 \exp(\lambda_1 t)\,\mathbf{V_1} + c_2 \exp(\lambda_2 t)\,\mathbf{V_2}$

7) Given an initial condition (x0, y0) solve $X(0) = c_1\,\mathbf{V_1} + c_2\,\mathbf{V_2}$ for c1, c2 and draw the trajectory.
(This is the full solution)

Tricky points

- Trajectories never cross: don’t draw them crossing on a phase portrait, even roughly (they meet at nodes and stars, but not at saddlepoints)
- Nullclines are curves defined by dx/dt = 0 or dy/dt = 0, they are not always a trajectory of the system.
- Nullclines do not always coincide with the axes
- Nullclines are straight lines for linear 2D systems (why?), but are usually curves for non-linear systems.
- Be able to distinguish slow/fast eigenvectors of a node from stable/unstable manifolds of a saddlepoint and nullclines
- Know the tau-delta plot. The type of fixed points on the boundaries between regions may not be correctly identified by linear stability analysis for nonlinear systems.