Recap of Lecture 1

Recipe for any 1D, autonomous non-linear ODE

1) Given the 1D equation: $dx/dt = f(x)$, draw the graph of $f(x)$ against x (**Graph 1**)

2) Find the points x^* where $f(x^*) = 0$ (i.e., crosses the X axis), these are the fixed points $\{x^*\}$ of the system

3) Mark the direction of flow of the vector field dx/dt with arrows on the axes in all relevant regions

4) For each fixed point, x^*, evaluate $df/dx(x^*)$, and classify the stability according to the **sign** of $df(x^*)/dx$. Compare with the graphical method of the flow of the vector field.

5) Draw some typical trajectories starting from various values of x_0 (**Graph 2**)

6) Integrate the equation explicitly if possible

Linear Stability Analysis

If x^* is a fixed point of the ODE $dx/dt = f(x)$, i.e., $f(x^*) = 0$.

Let $\eta = x - x^*$

be a small quantity, so we are near the fixed point.

Make a Taylor expansion in η around x^*

$d\eta/dt = dx/dt = f(x) = f(x^* + \eta) \sim f(x^*) + \eta\, df/dx(x^*) + \ldots$

$$d\eta/dt = \eta\, f'(x^*),$$

which has the exponential solution: $\eta(t) = \eta_0 \exp(f'(x^*)\, t)$

FP stability depends only on the sign of the first derivative at the fixed point $f'(x^*)$:

Stable fixed point ($df/dx < 0$)

Unstable fixed point ($df/dx > 0$)

Semi-stable fixed point (if $df/dx = 0$, look at higher-order terms)

There is only one 1st order **linear** ODE, the two parameters just rescale the x and t axes.

$dx/dt = ax + b$ a, b are real constants with units 1 / time and [x] / time respectively

Let $y = x + b/a$, and $\tau = at$, then this becomes

$$dy/d\tau = y$$

This is called non-dimensionalising the equation.

Equations with many parameters are hard to understand, so we try to reduce their number by non-dimensionalising. If we can eliminate a parameter it, it tells us 3 things:

1) the parameter is not really important; it doesn't determine the *shape* of the equation, only specific details, like a time scale.

2) Removing a parameter makes it easier to understand the system, because we can explore how the behaviour changes when we change the remaining parameters. This is much harder when there are 3, 4, or 5 parameters.

3) Fewer parameters = fewer mistakes in algebra.

Lecture 1: Difficulties? Silly mistakes? Tricky points?

The final slide in the introductory slides will be a list of "tricky" points.

These are NOT the only important points in a lecture, but experience has shown that people forget them in the exam.

- Be able to relate Graph 1 to Graph 2, i.e., go from the vector field to trajectories
- Trajectories never cross (but do meet at nodes) in any D
- Trajectories are monotonic in 1D
- Adjacent fixed points are of opposite stability
- What about adjacent semi-stable fixed points?

Lecture 2: Population growth is a big topic in dynamical systems

- We know some things about populations from intuition:

  Let N = # of individuals in a species (rabbits, bacteria, etc)

  NB. $N = 0$ is always a fixed point (no individuals $\Rightarrow$ no reproduction, in the absence of migration or spontaneous generation)

  dN/dt near $N \sim 0$ is probably linear in N (why would it be anything else?), so we write:

  $dN/dt = N\,R(N)$, and focus on the dynamics of $R(N)$

- This equation has a shape that tells us about the long-time size of a population. Its fixed points are a first approximation to robustness in living systems: start in many places, end up in one

Background quiz

Background quiz: go.epfl.ch/turningpoint

Session Id: julian23

All input is anonymous; data are stored outside CH

Break

Swiss Study Foundation

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Tricky points

- For autonomous ODEs like $dx/dt = F(x)$, the vector field dx/dt depends only on the value of x NOT the time when the system arrives at x nor the initial condition x_0
- Population models live in the first quadrant only
- To model a dynamical system, we need to find an equation with the same *qualitative* behaviour as the real system. This is called being in the same "universality class."

  e.g., There is no *qualitative* change in the population dynamics equation if we change the Logistic equation to: $dN/dt = r N^2(1 - N / K)$? Only the *detailed* time-dependence will be different. (*this is not really tricky, just a reminder*)