Graded exercise 2

- Graded Exercise 2 will be available on moodle after this lecture, and the solutions must be uploaded to moodle before midnight Tuesday, 26th November

- Submit early in case of glitches with the system

- Each person must submit their own solution, no photocopies, nor one person in a group submitting a single solution for several

- All questions must be done by hand (assigned points are similar to final exam)

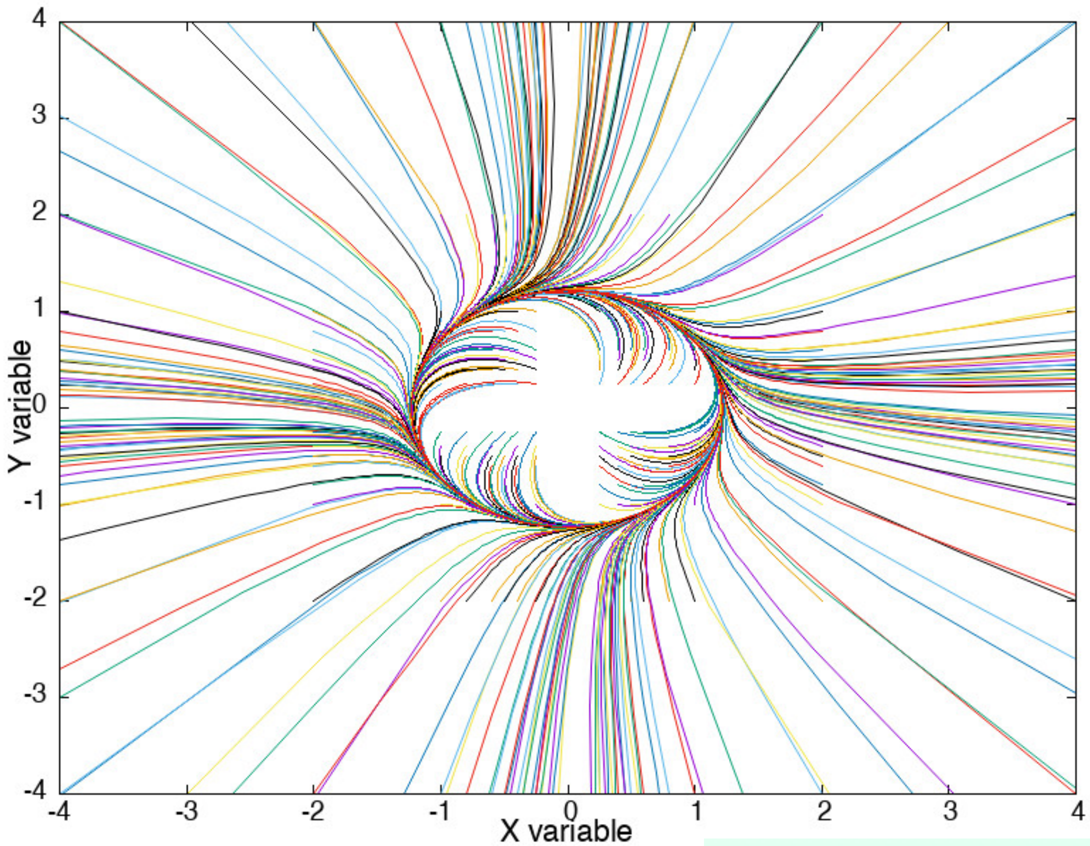
- Submit your work as a pdf file with your name/surname: **julian.shillcock.pdf**

Challenge: Can you find 2 limit cycles?

… starting from …

Edible prize

$$\frac{dx}{dt} = -y + x\,(1 - x^2 - y^2) + \ldots$$
$$\frac{dy}{dt} = x + y\,(1 - x^2 - y^2) + \ldots$$



First 3 people/groups who send me the phase portrait and the equations

I asked you to find an *example* of 2 limit cycles because …

https://en.wikipedia.org/wiki/Hilbert's_sixteenth_problem

The second part of Hilbert's 16th problem [edit] Unsolved

Here we are going to consider polynomial vector fields in the real plane, that is a system of differential equations of the form:

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y)$$

where both P and Q are real polynomials of degree n.

These polynomial vector fields were studied by Poincaré, who had the idea of abandoning the search for finding exact solutions to the system, and instead attempted to study the qualitative features of the collection of all possible solutions.

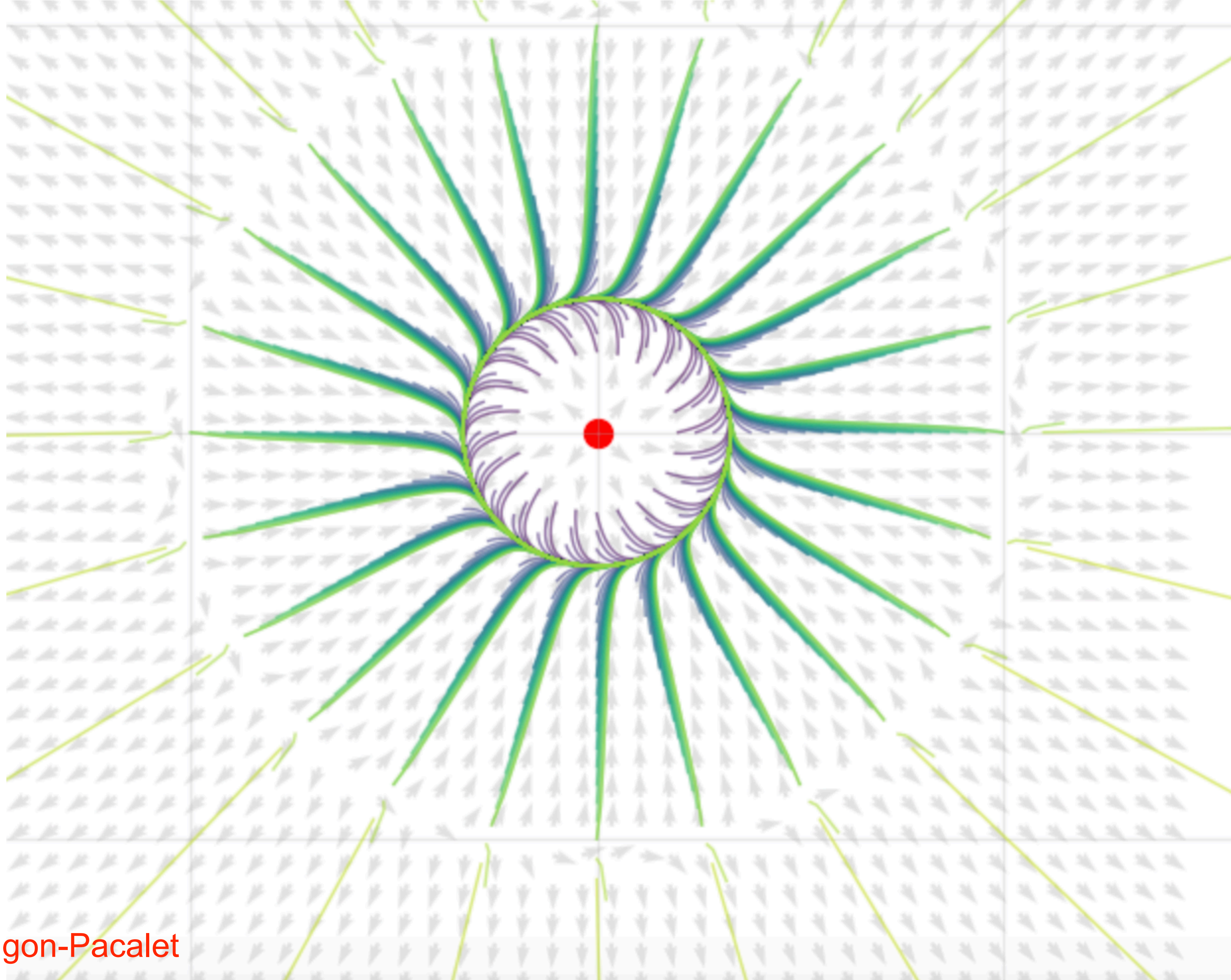
Among many important discoveries, he found that the limit sets of such solutions need not be a stationary point, but could rather be a periodic solution. Such solutions are called limit cycles.

The second part of Hilbert's 16th problem is to decide an upper bound for the number of limit cycles in polynomial vector fields of degree n and, similar to the first part, investigate their relative positions.

The question whether there exists a finite upper bound $H(n)$ for the number of limit cycles of planar polynomial vector fields of degree n remains unsolved for any $n > 1$. ($H(1) = 0$ since linear vector fields do not have limit cycles.) Evgenii Landis and Ivan Petrovsky claimed a solution in the 1950s, but it was shown wrong in the early 1960s. Quadratic plane vector fields with four limit cycles are known.[3] An example of numerical visualization of four limit cycles in a quadratic plane vector field can be found in.[4][5] In general, the difficulties in estimating the number of limit cycles by numerical integration are due to the nested limit cycles with very narrow regions of attraction, which are hidden attractors, and semi-stable limit cycles.

Equations of the 2 limit cycles:

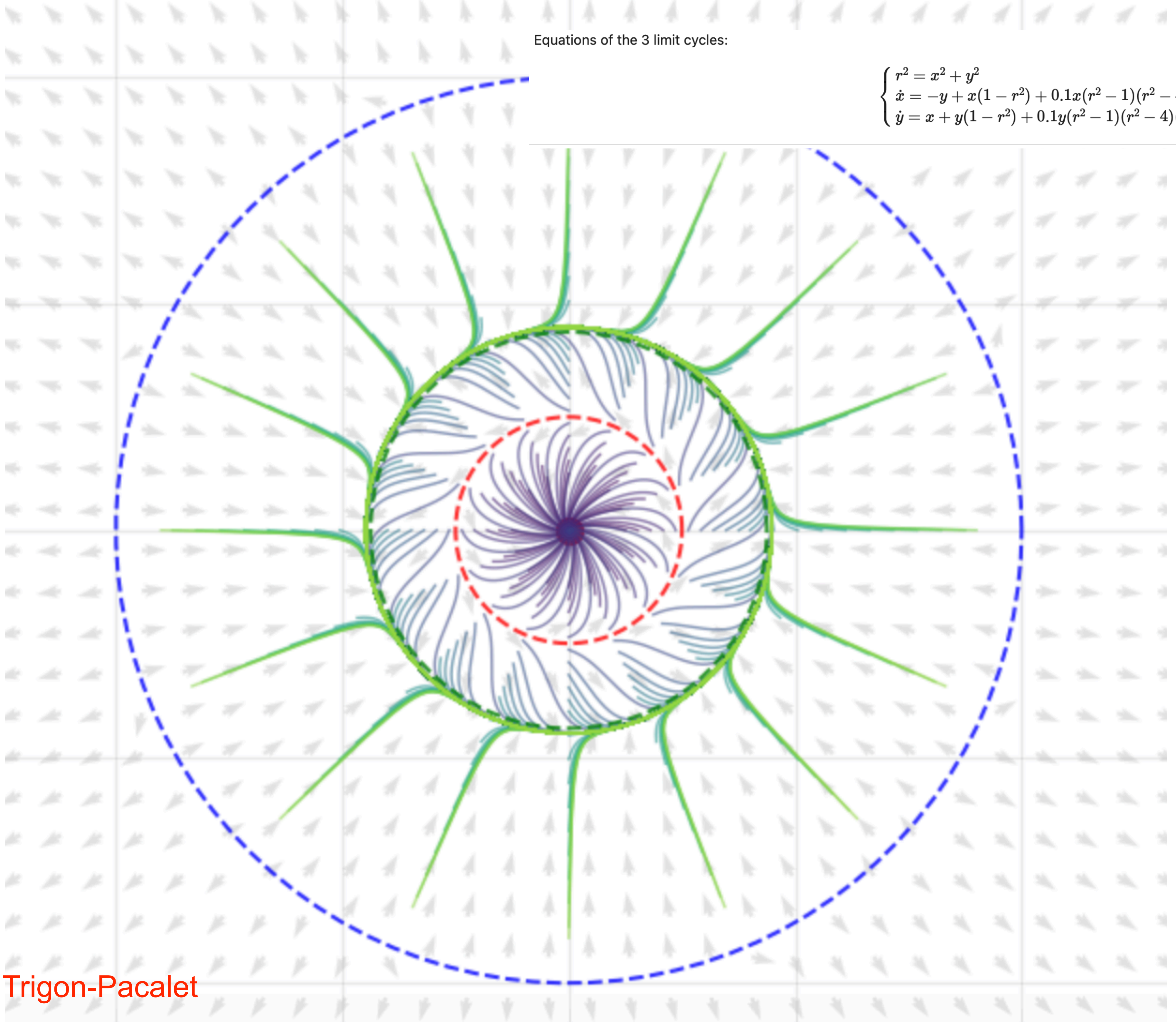
$$\begin{cases} r^2 = x^2 + y^2 \\ \dot{x} = -y + x(1 - r^2) + 0.1x(r^2 - 4)(r^2 - 16) \\ \dot{y} = x + y(1 - r^2) + 0.1y(r^2 - 4)(r^2 - 16) \end{cases}$$



Alice Trigon-Pacalet

Equations of the 3 limit cycles:

$$\begin{cases} r^2 = x^2 + y^2 \\ \dot{x} = -y + x(1 - r^2) + 0.1x(r^2 - 1)(r^2 - 4)(r^2 - 16) \\ \dot{y} = x + y(1 - r^2) + 0.1y(r^2 - 1)(r^2 - 4)(r^2 - 16) \end{cases}$$



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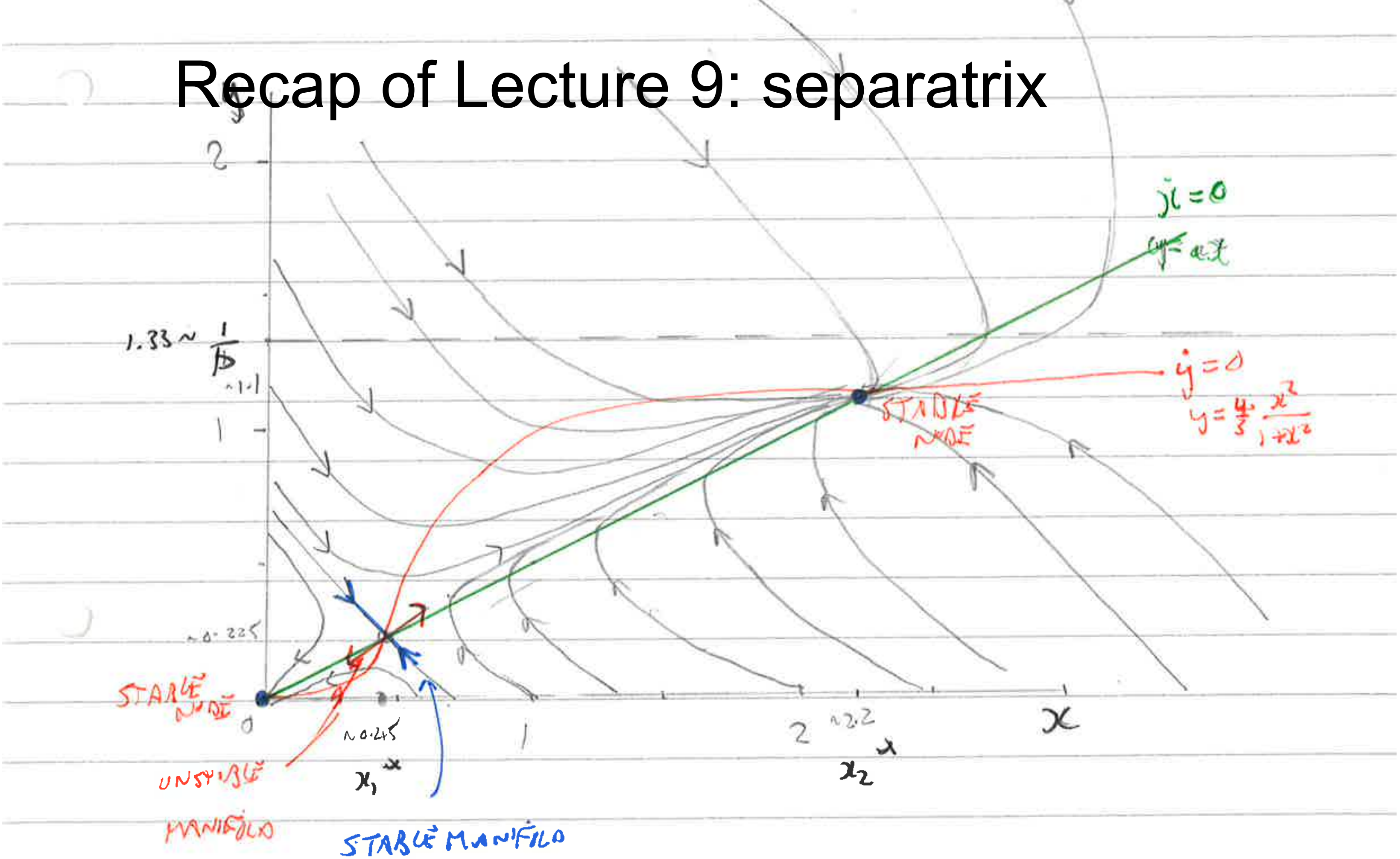
Recap of Lecture 9: separatrix

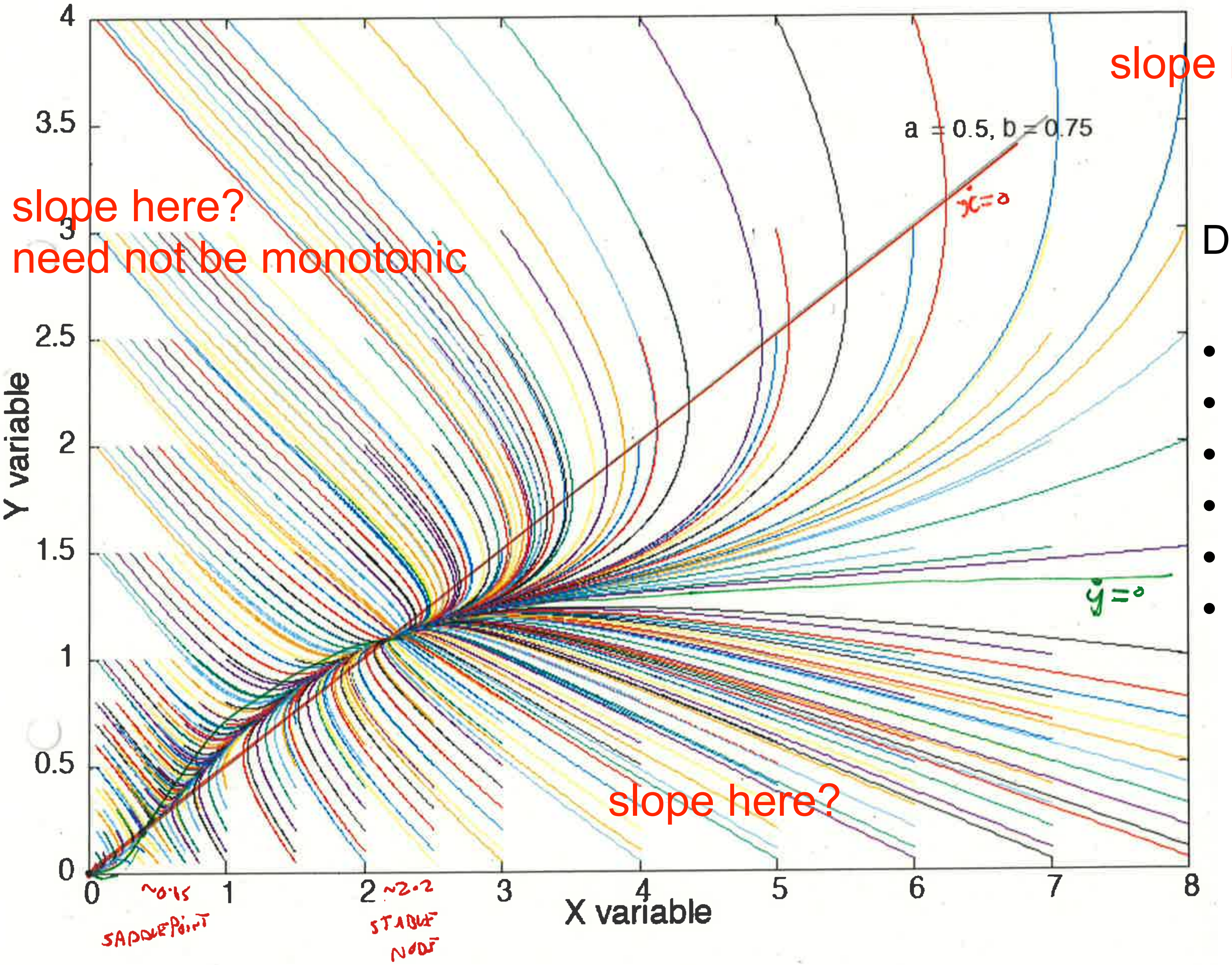


Difference between the unstable and stable manifolds of a saddlepoint:

Trajectories coming from initial points on opposite sides of the stable manifold go to different fixed points because they become parallel to the unstable manifold as time increases; hence the name “separatrix” because it separates the phase plane into discrete regions. NB Nullclines here are not trajectories.

Trajectories starting on opposite sides of the unstable manifold can go to the same fixed point (depending on other fixed points)

Recap of Lecture 9: drawing trajectories



Draw vector field on:

- nullclines
- axes
- near the axes
- fixed points
- large x OR y
- large x and Y

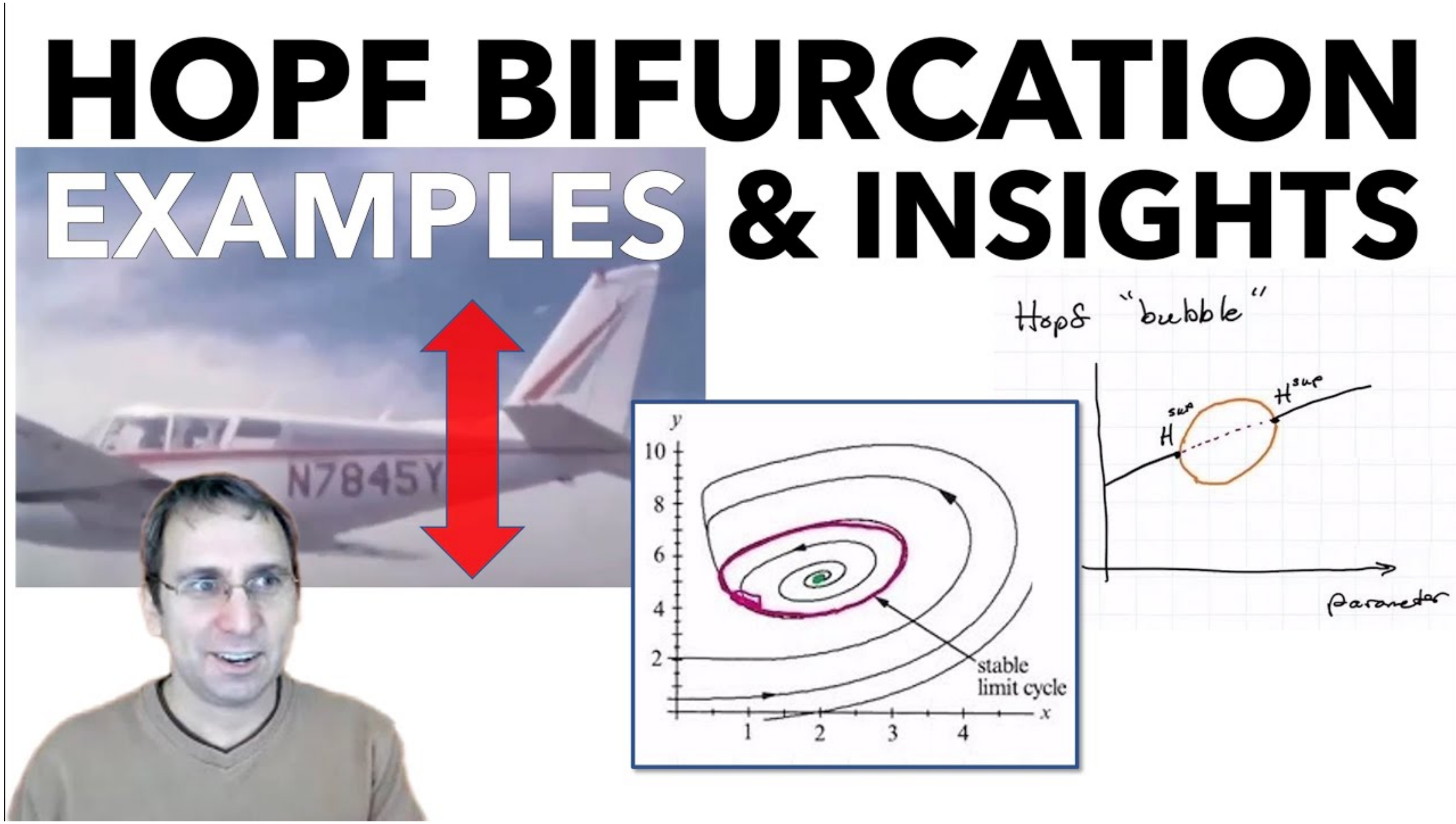
Background quiz: go.epfl.ch/turningpoint

Session Id: julian23

All input is anonymous; data are stored outside CH

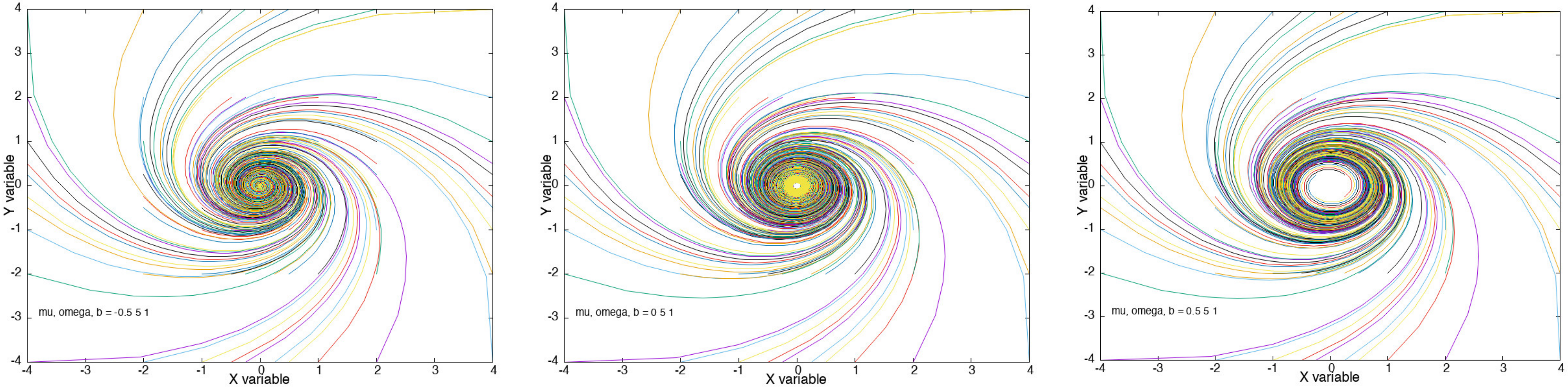
Break

Lecture 10 - Don't fly too fast …



https://www.youtube.com/watch?v=4vOC7zw2YME

Hopf bifurcation

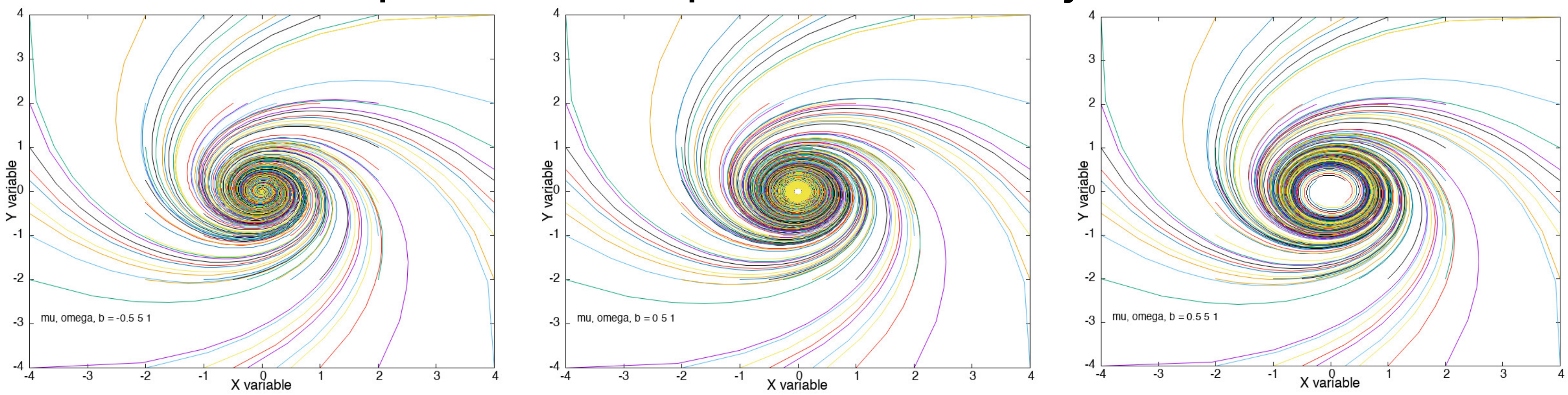


μ = -0.5 μ = 0 μ = 0.5

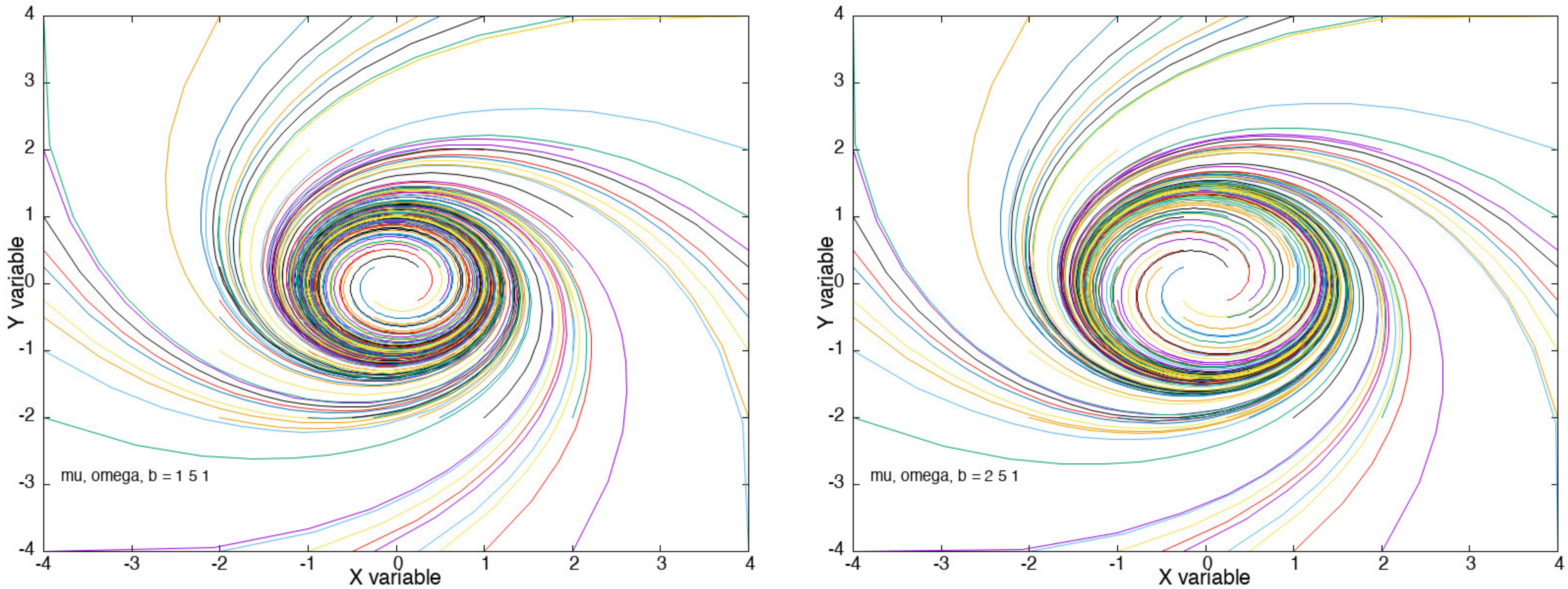
ω = 5, b = 1 for all plots

Important point: the bifurcation occurs at a specific value of μ, other things may occur when μ increases further. The stable spiral (μ < 0) becomes unstable (μ > 0), but if μ increases further, the unstable spiral may change again to be an unstable node or something else may happen: but it's still a Hopf bifurcation at μ = 0.

Supercritical Hopf bifurcation trajectories

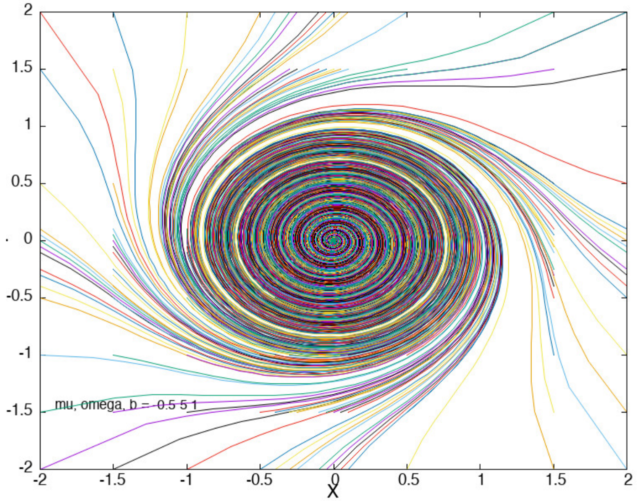


μ = -0.5 μ = 0 μ = 0.5

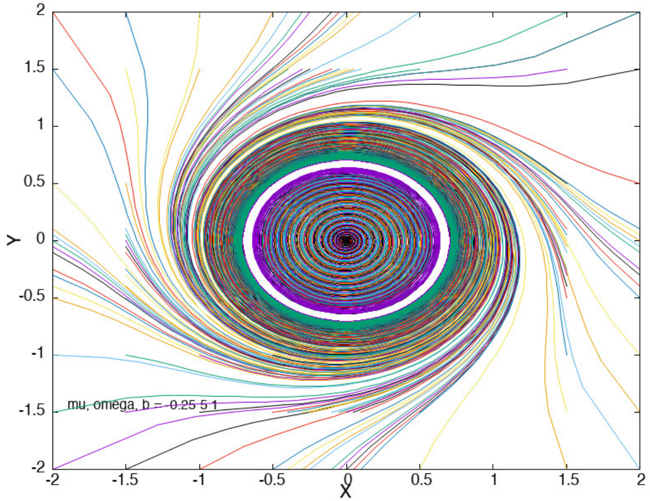


μ = 1 ω = 5, b = 1 for all plots μ = 2

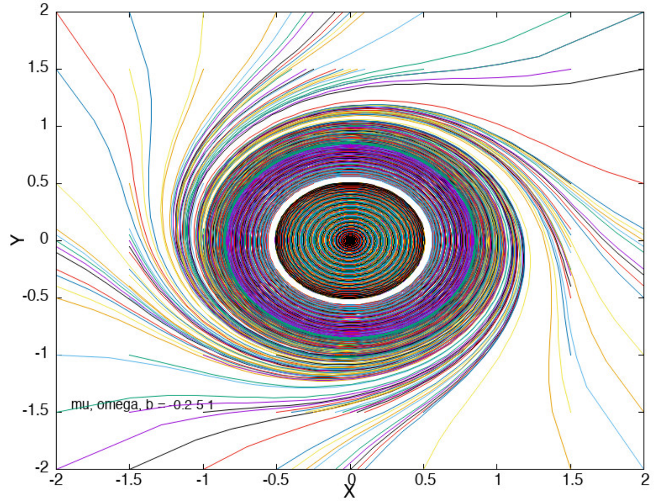
Subcritical Hopf bifurcation trajectories



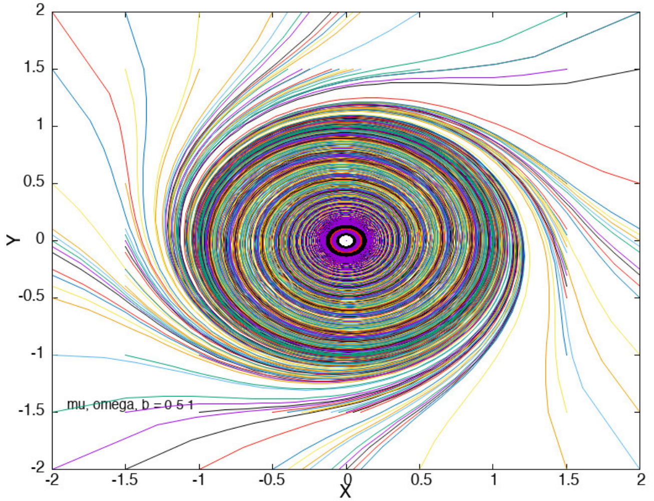
μ = -0.5



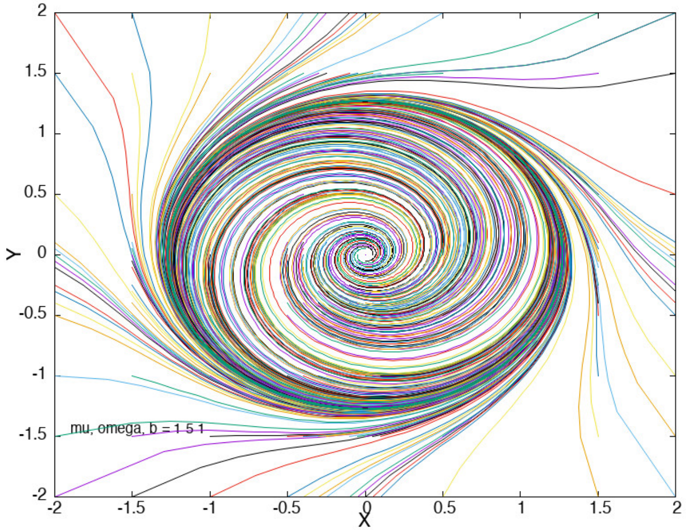
μ = -0.25



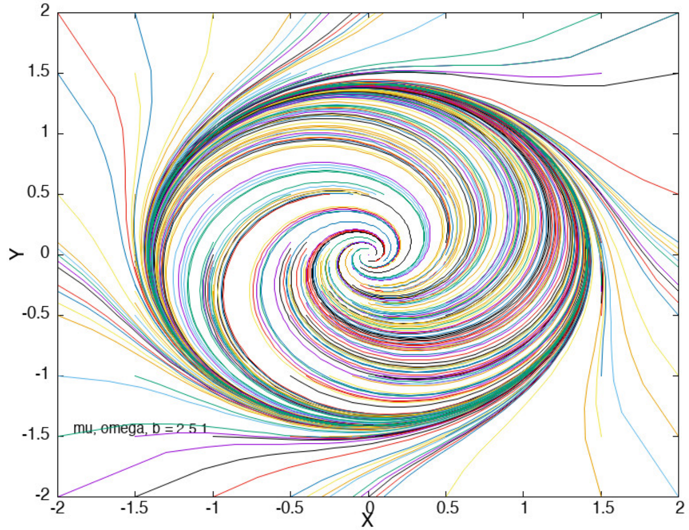
μ = -0.2



μ = 0

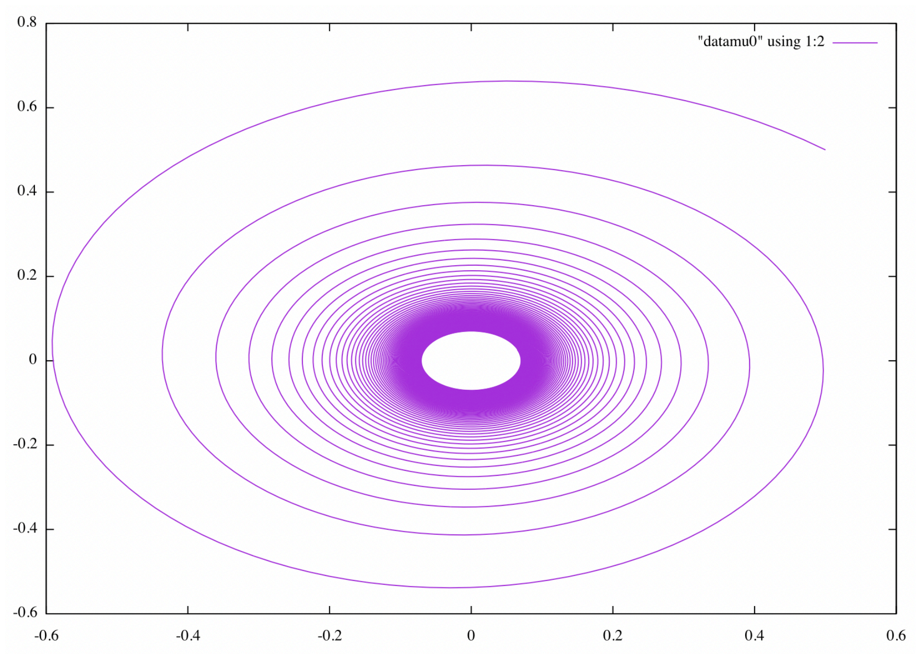


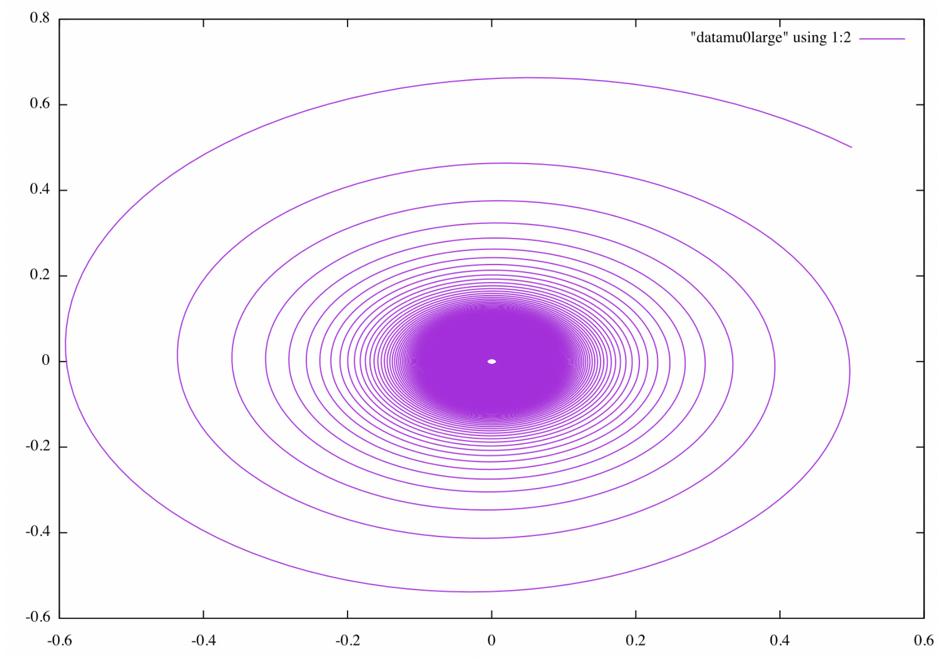
μ = 1



μ = 2

ω = 5, b = 1 for all plots





$\mu = 0, \omega = 5, b = 1$
N = 10,000 points

$\mu = 0, \omega = 5, b = 1$
N = 1000,000 points

Still a stable spiral at $\mu = 0$, the “hole” is an artifact of the number of integration steps.

$$dr/dt = \mu r - r^3$$
$$d\varphi/dt = \omega + b r^2$$

Strogatz, Ch. 8, page 253

“Tricky” points

- A bifurcation occurs at a *specific value* of a parameter; if something else happens as the parameter changes further, this doesn’t affect the type of bifurcation

- A Hopf bifurcation is a 2D analogue of the pitchfork bifurcation with an angular term added: hence a fixed point at non-zero x^* becomes a limit cycle at non-zero r^*

- Radius of a supercritical (soft, continuous, safe) Hopf bifurcation grows continuously from zero

- Radius of a subcritical (hard, discontinuous, dangerous) Hopf bifurcation starts at a large value

- Linear stability analysis cannot distinguish a supercritical from a subcritical Hopf bifurcation: it’s the nonlinear terms that do that