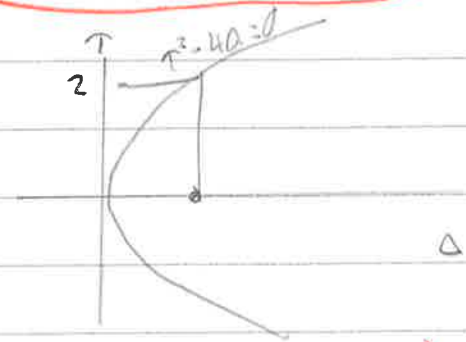
Difference between a star node and a degenerate node

24/10/24

Consider $\underset{\sim}{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$Tr\,\underset{\sim}{M} = 2$

$\Delta = 1$



$\therefore$ It is a star or degenerate node (unstable)

Eigenvalues

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)^2 = 0 \quad \therefore \lambda = 1$

Eigenvector

$\lambda = 1 \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \therefore \underline{v} = \begin{pmatrix} a \\ b \end{pmatrix}$: any vector in the plane

$\therefore$ It is an unstable star node

Now consider $\underline{M} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ i.e. $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$

$\operatorname{Tr} \underline{M} = 2$

$\Delta = 1$ again. So this is again a star or a degenerate node (unstable).

Eigenvalues

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

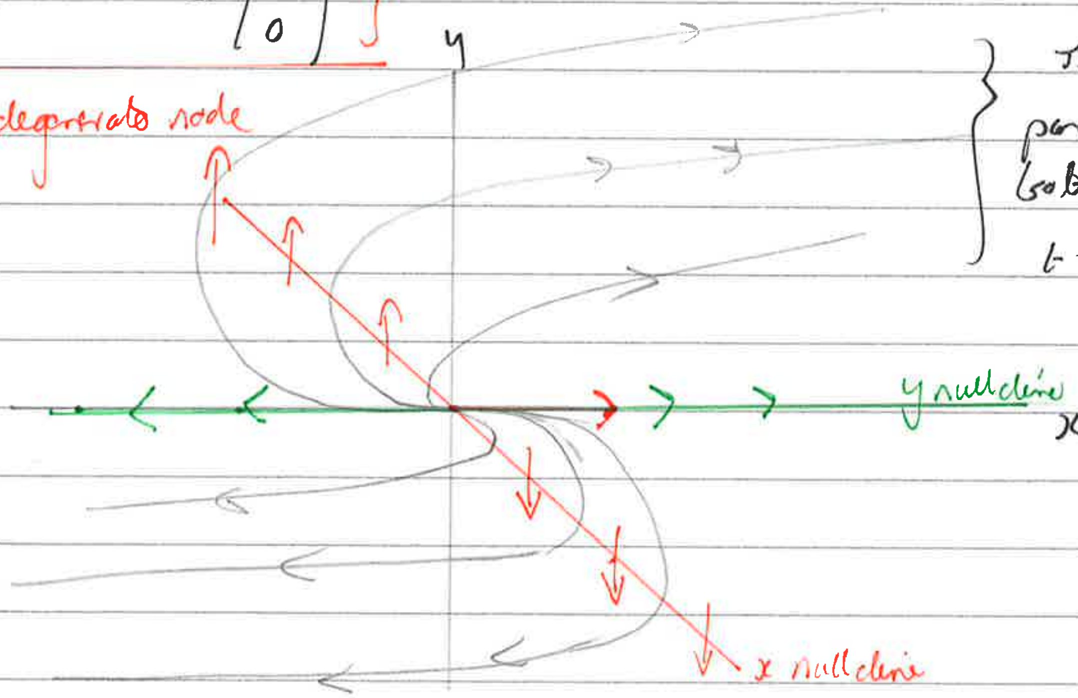
$\Rightarrow (1-\lambda)^2 = 0 \therefore \lambda = 1$ again

Eigenvector

$\lambda = 1$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \therefore v_2 = 0$ and eigenvector $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore \underline{x}(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ } incomplete solution

unstable, degenerate node



} Trajectories are parallel to eigenvector both as $t \to \infty$ and $t \to -\infty$.

Where are the null clines?

$\dot{x} = 0 \Rightarrow x + y = 0 \therefore y = -x$ is the x nullcline.

$\dot{y} = 0 \Rightarrow y = 0 \therefore y = 0$ is the y nullcline.

What is $\dot{y}$ along the x nullcline? $\dot{y} = y = -x < 0$ for $x > 0$

$\quad\quad > 0$ for $x < 0$.

and what is $\dot{x}$ along y nullcline? $\dot{x} = x + y = x > 0$ for $x > 0$

$\quad\quad < 0$ for $x < 0$.

Clearly $\underline{x}(t) = x_0 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not the full solution. We have for $\dot{y}$:

$$y(t) = y_0 e^t$$

and substituting this into the $\dot{x}$ equation:

$$\dot{x} = x + y_0 e^t$$

Rewrite this as: $\dot{x} - x = y_0 e^t$

cp. $y' + P(x)y = Q(x)$ ODE
see Handbook of Differential Eqn.
p 259

we solve this using an Integrating factor.

$I.F. = e^{-t}$ and multiply the equation by this:

$$\Rightarrow (\dot{x} - x) e^{-t} = y_0$$

$$\Rightarrow \frac{d}{ds}\left(e^{-t}x\right) = y_0$$

$$\therefore e^{-t}x = y_0 t + x_0$$

$$\therefore x(t) = (x_0 + y_0 t)e^{+t}$$

and the full solution is: $\underset{\sim}{x}(t) = \begin{pmatrix} (x_0+y_0t)e^{t} \\ y_0 e^{t} \end{pmatrix}$

check

1) $t=0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ ✓

2) $y_0 = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xe^{t} \\ 0 \end{pmatrix}$ ✓

3) If $x_0 < 0$ and $y_0 > 0$ and $y_0 \leq |x_0|$ then the trajectory goes along the negative x direction until $x_0 + y_0 t > 0$, when it turns and goes along positive x direction.

note that $y(t)$ monotonically increases with time.