Relativity and Cosmology I

Written exam - 16/01/2023

Some advice:

- You can use a formulaire (max 4 pages and readable without magnifying glasses).
- You may use formulas explained in the lectures without deriving them, as long as you make your reasoning clear.
- Make sure your final result is not absurd (we subtract up to 50% of the mark in this case). A final result can be absurd for several reasons: it is dimensionally incorrect, it is incompatible with an obvious symmetry or conservation law, it goes against basic physical intuition, etc.
- Write your answers with a pen and with clear handwriting.

1. Surface

Consider a two dimensional surface with the following metric

$$ds^2 = \left(1 + \frac{r^2}{a^2}\right) \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2 \,, \tag{1}$$

with $r > 0$ and $\phi \in [0, 2\pi]$.

(a) What is the length of the curve C defined by the equation $r = R$?

(b) What is the area inside the curve C?

(c) What is the distance from the point $r = 0$ to the curve C?
Hint: the following integral may be useful $\int dx \cosh^2 x = \frac{x}{2} + \frac{1}{4}\sinh(2x)$.

(d) Show that this surface can be isometrically embedded as a paraboloid in $\mathbb{R}^3$.

2. Black Hole in 2+1 dimensions

Consider the following metric

$$ds^2 = -f(r)\mathrm{d}t^2 + g(r)\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \,. \tag{2}$$

The independent and non-vanishing Christoffel symbols and Riemann tensor components **are given below**.

(a) Evaluate the Ricci tensor and Ricci scalar for this metric.

(b) Verify that

$$f(r) = \frac{1}{g(r)} = \frac{r^2 - r_h^2}{l^2} \,, \tag{3}$$

with r_h and l constants, solves the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \,. \tag{4}$$

Compute the energy-momentum tensor $T_{\mu\nu}$ and comment on its possible origin.

(c) Consider timelike and null geodesics for this metric with (3). Denote by E and L energy and angular momentum, i.e. the conserved quantities corresponding to the cyclic variables of this metric. Show that the radial equation can be recasted as

$$\frac{\dot{r}^2}{2} + V(r) = \frac{E^2}{2} \,. \tag{5}$$

Find $V(r)$ for timelike and null geodesics.

(d) Are there circular orbits in this spacetime? Consider the null and timelike cases separately. Explain your answer.

(e) What is the maximal proper time that an unaccelerated observer can remain outside the event horizon at $r = r_h$?

Independent non-vanishing components of the Christoffel symbols

$$\Gamma^{\phi}_{\phi r} = \frac{1}{r}\,, \qquad \Gamma^{t}_{tr} = \frac{f'}{2f}\,, \qquad \Gamma^{r}_{\phi\phi} = -\frac{r}{g}\,, \qquad \Gamma^{r}_{tt} = \frac{f'}{2g}\,, \qquad \Gamma^{r}_{rr} = \frac{g'}{2g}\,. \tag{6}$$

Independent non-vanishing components of the Riemann tensor

$$R^{\phi}{}_{t\phi t} = \frac{f'}{2rg}\,, \qquad R^{\phi}_{r\phi r} = \frac{g'}{2rg}\,, \qquad R^{t}_{rtr} = \frac{1}{4f^2 g}\left(ff'g' + g(f'^2 - 2ff'')\right)\,. \tag{7}$$

3. Explosion

Consider an isolated body at rest in outer space. The body contains a bomb that explodes suddenly and breaks the body into n pieces that fly away with non-relativistic velocities $\vec{v}_a$ where $a \in \{1, \ldots, n\}$ labels each piece.

(a) Compute the quadrupole moment I_{ij} of the system before and after the explosion.

(b) Compute the second time derivative of the quadrupole moment $\ddot{I}_{ij}$ after the explosion and show that it is independent of the origin of the coordinate system.

(c) What is the maximum possible value of $\ddot{I}^2 \equiv \ddot{I}_{ij}\ddot{I}_{kl}\delta^{ik}\delta^{jl}$, knowing that the explosion released an energy E? For this question, you can assume that all the energy is converted into kinetic energy. What explosion pattern maximizes $\ddot{I}^2$?

(d) Assuming the explosion lasted for a time T, estimate the order of magnitude of the energy emitted in gravitational waves. What do you get for a nuclear explosion with $E = 25$ megatons of TNT $\approx 10^{17} J$ and $T \sim 10^{-6} s$?

$G = 6.67 \times 10^{-11}\, m^3 kg^{-1} s^{-2}$
$c = 3 \times 10^{8}\, m/s$
$\hbar = 1.05 \times 10^{-34}\, m^2 kg/s$