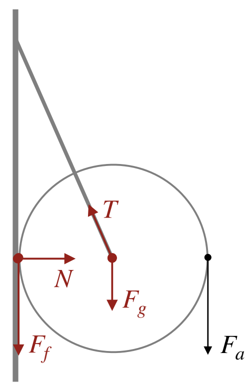
Solution to the Final Exam

20 January 2023

PHYS-101(en)

1. Toilet paper

a. The free body diagram for the roll of toilet paper is shown in the figure below.



b. The torque $\vec{\tau}$ arising from each force is calculated from the formula

$$\vec{\tau} = \vec{R} \times \vec{F}, \tag{1}$$

where $\vec{R}$ is the position vector from the pivot point G to the point of application of the force. Plugging in the values for each force gives

$$\vec{\tau}_g = \vec{R}_g \times \vec{F}_g = 0 \times \vec{F}_g = 0 \tag{2}$$

$$\vec{\tau}_N = \vec{R}_N \times \vec{N} = -R\hat{\imath} \times N\hat{\imath} = 0 \tag{3}$$

$$\vec{\tau}_f = \vec{R}_f \times \vec{F}_f = -R\hat{\imath} \times F_f\hat{\jmath} = -RF_f\hat{k} \tag{4}$$

$$\vec{\tau}_T = \vec{R}_T \times \vec{T} = 0 \times \vec{T} = 0 \tag{5}$$

$$\vec{\tau}_a = \vec{R}_a \times \vec{F}_a = R\hat{\imath} \times F_a\hat{\jmath} = RF_a\hat{k}. \tag{6}$$

c. When F_a is weak, the static friction force between the roll and the wall will be sufficient to prevent the roll from rotating. However, as you pull harder (corresponding to a larger F_a) you will eventually exceed the maximum possible static friction force and the roll will start to rotate. Thus, we must calculate the maximum value of the static friction force and also determine the strength of the friction force F_s required to counteract a given applied force F_a.

The maximum value of the static friction force is given by

$$F_f \leq \mu_s N, \tag{7}$$

where we still must determine the normal force N. The normal force and the connection between F_s and F_a can be found by enforcing the conditions required for static equilibrium,

$$\sum \vec{\tau} = 0 \tag{8}$$

and

$$\sum \vec{F} = 0. \tag{9}$$

Substituting equations (2) through (6) into equation (8) gives

$$0 + 0 - RF_f\hat{k} + 0 + RF_a\hat{k} = 0 \quad \Rightarrow \quad F_a = F_f, \tag{10}$$

where we have taken the $\hat{k}$ component of the equation.

From the free body diagram in part a, we see that equation (9) is

$$\vec{F}_a + \vec{F}_g + \vec{F}_f + \vec{N} + \vec{T} = 0 \quad \Rightarrow \quad F_a\hat{\jmath} + Mg\hat{\jmath} + F_f\hat{\jmath} + N\hat{\imath} - T\left(\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}\right) = 0. \tag{11}$$

Taking the $\hat{\imath}$ component gives

$$N - T\sin\theta = 0 \quad \Rightarrow \quad N = T\sin\theta, \tag{12}$$

while the $\hat{\jmath}$ component is

$$F_a + Mg + F_f - T\cos\theta = 0. \tag{13}$$

To find the maximum possible value of F_a without the roll rotating, we let equation (7) be an equality to get

$$F_f = \mu_s N. \tag{14}$$

Plugging this and equation (12) into equation (10) gives

$$F_a = \mu_s N = \mu_s T\sin\theta \quad \Rightarrow \quad T = \frac{F_a}{\mu_s\sin\theta}. \tag{15}$$

Substituting equations (12), (14), and (15) into equation (13) gives

$$F_a + Mg + \mu_s\left(T\sin\theta\right) - T\cos\theta = 0 \quad \Rightarrow \quad F_a + Mg + \left(\mu_s\sin\theta - \cos\theta\right)\frac{F_a}{\mu_s\sin\theta} = 0. \tag{16}$$

Solving for F_a gives the final answer of

$$F_a\mu_s\sin\theta + \left(\mu_s\sin\theta - \cos\theta\right)F_a = -Mg\mu_s\sin\theta \quad \Rightarrow \quad F_a\left(\cos\theta - 2\mu_s\sin\theta\right) = Mg\mu_s\sin\theta \tag{17}$$

$$\Rightarrow \quad F_a = \frac{\mu_s\tan\theta}{1 - 2\mu_s\tan\theta}Mg. \tag{18}$$

This can also be written as

$$F_a = \frac{\mu_s}{2}\left(\frac{1}{2\tan\theta} - \mu_s\right)^{-1}Mg. \tag{19}$$

Expressed in this form, we clearly see the reason for the condition given in the problem statement that $1/(2\tan\theta) > \mu_s$. If this wasn't the case, the value we found for the magnitude of the applied force F_a would diverge or be negative.

d. When the roll is rotating, the net torque no longer has to be zero as the roll can have a non-zero angular acceleration. However, we know that the center of mass of the roll is fixed by the metal rods and the contact with the wall. To calculate the force that the rods must exert on the roll to keep it in place, we enforce

$$\sum \vec{F} = 0. \tag{20}$$

As in part c, the $\hat{\imath}$ component is

$$N - T\sin\theta = 0 \quad \Rightarrow \quad N = T\sin\theta \tag{21}$$

and the $\hat{\jmath}$ component is

$$F_a + Mg + F_f - T\cos\theta = 0. \tag{22}$$

The difference with part c arises from the frictional force, which is now kinetic and is given by

$$F_f = \mu_c N. \tag{23}$$

Substituting this and equation (21) into equation (22) gives

$$F_a + Mg + \mu_c\left(T\sin\theta\right) - T\cos\theta = 0 \quad \Rightarrow \quad F_a + Mg + T\left(\mu_c\sin\theta - \cos\theta\right) = 0. \tag{24}$$

Solving for T, we find

$$T = \frac{1}{\cos\theta - \mu_c\sin\theta}\left(F_a + Mg\right). \tag{25}$$

Substituting this into equation (21) gives

$$N = \frac{\sin\theta}{\cos\theta - \mu_c\sin\theta}\left(F_a + Mg\right). \tag{26}$$

However, the problem specifically asks for the vector expressions, not just the magnitudes of the forces. From the corresponding terms appearing in equation (11) (or the free body diagram in part a), we see that the final answers are

$$\vec{T} = -\frac{\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}}{\cos\theta - \mu_c\sin\theta}\left(F_a + Mg\right) \tag{27}$$

and

$$\vec{N} = \frac{\sin\theta\hat{\imath}}{\cos\theta - \mu_c\sin\theta}\left(F_a + Mg\right). \tag{28}$$

These expressions can also be written as

$$\vec{T} = -\left(\hat{\imath} + \frac{1}{\tan\theta}\hat{\jmath}\right)\left(\frac{1}{\tan\theta} - \mu_c\right)^{-1}\left(F_a + Mg\right) \tag{29}$$

and

$$\vec{N} = \left(\frac{1}{\tan\theta} - \mu_c\right)^{-1}\left(F_a + Mg\right)\hat{\imath}. \tag{30}$$

Thus, since $1/(2\tan\theta) > \mu_c$, $1/\tan\theta > 1/(2\tan\theta)$, and $\tan\theta > 0$ (as we know from physical intuition and the diagram in the problem statement that $0 < \theta < \pi/2$), all of the components of $\vec{T}$ and $\vec{N}$ are in the expected directions and do not diverge.

e. To determine the angular acceleration from the torque arising from the applied force $\vec{F}_a$, we can use Newton's second law for rotation

$$\sum \vec{\tau} = I_G\vec{\alpha}. \tag{31}$$

Substituting equations (2) through (6) into equation (31) gives

$$0 + 0 - RF_f\hat{k} + 0 + RF_a\hat{k} = I_G\vec{\alpha} \quad \Rightarrow \quad R\left(F_a - F_f\right)\hat{k} = I_G\alpha\hat{k} \quad \Rightarrow \quad \alpha = \frac{R}{I_G}\left(F_a - F_f\right), \tag{32}$$

where we have used $\vec{\alpha} = \alpha\hat{k}$ from the problem statement and taken the $\hat{k}$ component of the equation. Substituting equations (23) and (26) gives the final answer of

$$\alpha = \frac{R}{I_G}\left(F_a - \mu_c N\right) = \frac{R}{I_G}\left(F_a - \frac{\mu_c \sin\theta}{\cos\theta - \mu_c \sin\theta}\left(F_a + Mg\right)\right), \tag{33}$$

which is indeed a constant.

f. The situation of toilet paper coming off of a roll is similar to the no-slip condition for a rope moving over a pulley. From this we know that the linear speed of the end of the paper as it moves downwards must be equal to the tangential velocity of a point on the outer edge of the roll as it rotates around its center of mass, i.e.

$$\frac{dy}{dt} = R\omega. \tag{34}$$

This is the key condition that we will use to relate the rotational motion to the linear motion of the end of the paper.

The question tells us that we can treat the angular acceleration α as known and constant and the problem statement gives the formulas

$$\alpha = \frac{d^2\phi}{dt^2} \tag{35}$$

and

$$\omega = \frac{d\phi}{dt}. \tag{36}$$

Combining equations (35) and (36) gives

$$\alpha = \frac{d\omega}{dt}. \tag{37}$$

Integrating once in time yields

$$\omega = \alpha t + C_1. \tag{38}$$

Substituting this into equation (34) gives

$$\frac{dy}{dt} = R\alpha t + RC_1. \tag{39}$$

Using the initial condition on the velocity that $dy/dt = 0$ at $t = 0$, we find that $C_1 = 0$, so equation (39) becomes

$$\frac{dy}{dt} = R\alpha t. \tag{40}$$

Integrating once more in time yields

$$y(t) = \frac{R\alpha}{2}t^2 + C_2. \tag{41}$$

Applying the initial condition on the position that $y(0) = \ell$ allows us to find $C_2 = \ell$, which shows that the final answer is

$$y(t) = \frac{R\alpha}{2}t^2 + \ell. \tag{42}$$

2. DART mission

a. This problem is simpler than it initially appears. We know both the radius R_0 and the period T_0 of the circular orbit that D2 is executing about D1. Thus, D2 travels once around D1 in a time T_0 and travels a total distance of $2\pi R_0$. This means its average speed is

$$\overline{v}_0 = \frac{2\pi R_0}{T_0}. \tag{1}$$

Since there are no other forces in the problem, D2 must be executing uniform circular motion, so its average speed is identical to its instantaneous speed

$$v_0 = \frac{2\pi R_0}{T_0}. \tag{2}$$

Since the problem asks for the *velocity*, so we must determine the direction as well. From the diagram in the problem statement, we see that D2 moves purely in the $\hat{\phi}$ direction so

$$\vec{v}_0 = \frac{2\pi R_0}{T_0}\hat{\phi}. \tag{3}$$

b. In this part, we apply Newton's second law as it will enable us to find the mass from the force and the trajectory of the object (i.e. the acceleration). Newton's second law for the general gravitational force is given by

$$\vec{F}_G = m_2\vec{a} \quad \Rightarrow \quad -G\frac{m_1 m_2}{R_0^2}\hat{r} = -m_2\frac{v_0^2}{R_0}\hat{r}, \tag{4}$$

where we know from part a that D2 exhibits uniform circular motion and as such must be experiencing centripetal acceleration $\vec{a} = -v_0^2/R_0\hat{r}$. Taking the $\hat{r}$ component of equation (4) and solving for m_1 gives

$$m_1 = \frac{R_0}{G}v_0^2. \tag{5}$$

Lastly, we substitute (3) to get the final answer of

$$m_1 = \frac{4\pi^2}{G}\frac{R_0^3}{T_0^2}. \tag{6}$$

c. Using the definition of the density and the formula for the volume of a sphere, we can write the density of D1 as

$$\rho_1 = \frac{m_1}{4\pi r_1^3/3} \tag{7}$$

and the density of D2 as

$$\rho_2 = \frac{m_2}{4\pi r_2^3/3}. \tag{8}$$

Since we can assume the densities of the two are equal $\rho_1 = \rho_2$, we can equate these two equations and find

$$\frac{m_1}{4\pi r_1^3/3} = \frac{m_2}{4\pi r_2^3/3} \quad \Rightarrow \quad \frac{m_1}{r_1^3} = \frac{m_2}{r_2^3} \quad \Rightarrow \quad m_2 = \left(\frac{r_2}{r_1}\right)^3 m_1. \tag{9}$$

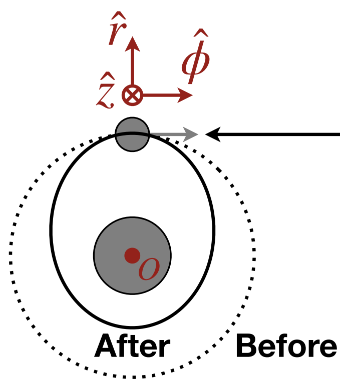
d. During an inelastic collision momentum is conserved, but mechanical energy is not. However, since the collision is *perfectly* inelastic we know that the objects stick together, meaning they have the same final velocity. Thus, conservation of momentum can be written as

$$m_2\vec{v}_0 + m_s\vec{v}_s = (m_2 + m_s)\,\vec{v}_a. \tag{10}$$

Given the directions of $\vec{v}_0$ and $\vec{v}_s$, we can simplify to find that the final answer is

$$m_2 v_0 \hat{\phi} - m_s v_s \hat{\phi} = (m_2 + m_s)\, \vec{v}_a \quad \Rightarrow \quad \vec{v}_a = \frac{m_2 v_0 - m_s v_s}{m_2 + m_s} \hat{\phi}. \tag{11}$$

e. The orbits before and after the collision are shown in the figure below. From Kepler's first law we know that the orbits of astronomical objects are ellipses. While the orbit before is circular (i.e. an ellipse with equal major and minor axes), the orbit after must be a non-circular ellipse. This is because just after the collision the inwards gravitational force remains unchanged. However, the collision reduces the velocity of D2, thereby decreasing the centripetal acceleration needed for uniform circular motion (i.e. $a_{cent} = v_0^2/R_0$). Thus, the gravitational force is larger than needed for uniform circular motion, so it pulls the object inwards.



f. We can still apply conservation of momentum as we did for equation (10), but the right-hand side must be modified to account for the ejected material. It becomes

$$m_2 \vec{v}_0 + m_s \vec{v}_s = (m_2 + m_s - m_e)\, \vec{v}_b + m_e \vec{v}_e. \tag{12}$$

Given the directions of $\vec{v}_0$, $\vec{v}_s$, and $\vec{v}_e$, we can simplify to find that the final answer is

$$m_2 v_0 \hat{\phi} - m_s v_s \hat{\phi} = (m_2 + m_s - m_e)\, \vec{v}_b + m_e v_e \hat{\phi} \quad \Rightarrow \quad \vec{v}_b = \frac{m_2 v_0 - m_s v_s - m_e v_e}{m_2 + m_s - m_e} \hat{\phi}. \tag{13}$$

g. Kepler's third law can be applied both before and after the collision. However, there are two versions of Kepler's third law, which are equivalent. The version (given in the problem statement) uses the major axis of the ellipse as the distance. Thus, from our diagram in figure e, we see that this version of Kepler's third law is given by

$$T_0^2 = C_1 \left(2R_0\right)^3 \tag{14}$$

before the collision, and

$$T_2^2 = C_1 \left(R_0 + R_2\right)^3 \tag{15}$$

after the collision. Note that the constant of proportionality C_1 is the same in these two equations. The other version of Kepler's third law (which we saw in class) uses the average radius (or more precisely the length of the semi-major axis) as the distance. Thus, from our diagram in figure e, we see that this version of Kepler's third law is given by

$$T_0^2 = C_2 R_0^3 \tag{16}$$

before the collision, and

$$T_2^2 = C_2 \left(\frac{R_0 + R_2}{2}\right)^3 \tag{17}$$

after the collision. Again the constant of proportionality C_2 is the same in these two equations, but it is different than C_1. However, the only difference between these two versions of Kepler's third law is whether a factor of 2^3 is included in the constant of proportionality or not. In class we saw that

$$C_2 = \frac{4\pi^2}{Gm_1}, \tag{18}$$

so the formulas are equivalent because

$$C_1 = \frac{1}{2^3}\frac{4\pi^2}{Gm_1}. \tag{19}$$

Thus, we can either combine equations (14) and (15) to eliminate C_1 or equations (16) and (17) to eliminate C_2. Either way we find the same answer of

$$\frac{T_0^2}{R_0^3} = 2^3 \frac{T_2^2}{(R_0 + R_2)^3}. \tag{20}$$

Solving for R_2 gives the final answer of

$$(R_0 + R_2)^3 = \left(\frac{T_2}{T_0}\right)^2 (2R_0)^3 \quad \Rightarrow \quad R_2 = 2\left(\frac{T_2}{T_0}\right)^{2/3} R_0 - R_0. \tag{21}$$

h. This problem is more difficult. To calculate the velocity immediately after the collision from the measured properties of the modified orbit, we need to use all of the information we have available: R_0, R_2, m_1, and G. Immediately after the collision, D2 is at its furthest position away from D1 (i.e. it is at a distance of R_0). We also know the position of closest approach, which is a distance of R_2. We can relate the velocity of D2 at these two positions using conserved quantities. To determine what quantities are conserved in the motion of D2, we must consider the forces on it, which are only the gravitational force from D1. Thus, if we take the pivot point to be the center of D1, the gravitational force will not exert a torque (as the direction of the force has no component perpendicular to the displacement vector from the pivot point to D2). This means that the angular momentum of D2 about the center of D1 is conserved. Additionally, since there are no non-conservative forces doing work on D2, the mechanical energy of the system is also conserved. Note that non-conservative forces do work *during* the collision, but in this part we are only interested in the motion from just after the collision until D2 reaches its point of closest approach.

First, we will enforce conservation of angular momentum. Angular momentum is defined to be $\vec{L} = \vec{R} \times m\vec{v}$, where $\vec{R}$ is the position vector to the object from a pivot. The pivot is the center of D1, which is also the origin of our coordinate system. Thus, conservation of angular momentum $\vec{L}_i = \vec{L}_f$ can be written as

$$R_0\hat{r} \times (m_2 + m_s - m_e)\,\vec{v}_c = R_2\hat{r} \times (m_2 + m_s - m_e)\,\vec{v}_2 \quad \Rightarrow \quad R_0\hat{r} \times \vec{v}_c = R_2\hat{r} \times \vec{v}_2, \tag{22}$$

where $\vec{v}_2$ is the velocity of D2 at its point of closest approach. Both $\vec{v}_c$ and $\vec{v}_2$ are unknown, but we know that they both are in the $\hat{\phi}$ direction. Substituting this fact gives

$$v_c R_0 \hat{r} \times \hat{\phi} = v_2 R_2 \hat{r} \times \hat{\phi} \quad \Rightarrow \quad v_c R_0 \hat{z} = v_2 R_2 \hat{z} \quad \Rightarrow \quad v_2 = v_c \frac{R_0}{R_2}, \tag{23}$$

where we have take the $\hat{z}$ component of the equation.

To eliminate v_2 we will enforce conservation of mechanical energy. Since the problem statement tells us that D2 does not rotate, the only contributions are translational kinetic energy and potential energy. If you do not already know it, the general form of the gravitational potential energy can be calculated from the general gravitational force to be

$$U_G = -\int \vec{F}_G \cdot d\vec{\ell} = -\int \left(-G\frac{m_a m_b}{r^2}\hat{r}\right) \cdot d\vec{\ell} = G m_a m_b \int \frac{1}{r^2} dr = G m_a m_b \left(-\frac{1}{r}\right) = -\frac{G m_a m_b}{r}, \tag{24}$$

where r is the distance between any two masses m_a and m_b and we have taken the reference point for the potential energy to be $r = \infty$ such that the integration constant is zero. This enables us to write conservation of mechanical energy $E_{mi} = E_{mf}$ as

$$\frac{m_2 + m_s - m_e}{2} v_c^2 - \frac{G m_1 (m_2 + m_s - m_e)}{R_0} = \frac{m_2 + m_s - m_e}{2} v_2^2 - \frac{G m_1 (m_2 + m_s - m_e)}{R_2}. \tag{25}$$

Plugging in equation (23) and simplifying gives

$$\frac{1}{2} v_c^2 - \frac{G m_1}{R_0} = \frac{1}{2}\left(v_c \frac{R_0}{R_2}\right)^2 - \frac{G m_1}{R_2} \quad \Rightarrow \quad v_c^2 - v_c^2 \frac{R_0^2}{R_2^2} = 2\frac{G m_1}{R_0} - 2\frac{G m_1}{R_2} \tag{26}$$

$$\Rightarrow \quad v_c^2 \left(1 - \frac{R_0^2}{R_2^2}\right) = 2 G m_1 \left(\frac{1}{R_0} - \frac{1}{R_2}\right) \quad \Rightarrow \quad v_c^2 \left(\frac{R_2^2 - R_0^2}{R_2^2}\right) = 2 G m_1 \left(\frac{R_2 - R_0}{R_0 R_2}\right) \tag{27}$$

$$\Rightarrow \quad v_c^2 = 2 G m_1 \frac{R_2 - R_0}{R_0} \frac{R_2}{R_2^2 - R_0^2} \quad \Rightarrow \quad v_c^2 = 2 G m_1 \frac{R_2 - R_0}{R_0} \frac{R_2}{(R_2 - R_0)(R_2 + R_0)} \tag{28}$$

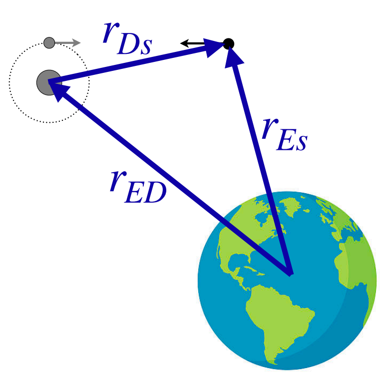
$$\Rightarrow \quad v_c = \pm\sqrt{\frac{2 G m_1}{R_0 + R_2} \frac{R_2}{R_0}}. \tag{29}$$

The problem asks for the velocity, so we consult the picture from part e and write

$$\vec{v}_c = \sqrt{\frac{2 G m_1}{R_0 + R_2} \frac{R_2}{R_0}}\,\hat{\phi}, \tag{30}$$

where we have chosen the positive sign based on the physical intuition that the small satellite won't completely reverse the direction of a huge asteroid.

i. We know that the satellite starts with an initial velocity of $\vec{v}_i$ in the reference frame of the Earth and ends with a final velocity of $\vec{v}_s$ in the reference frame of D1. To calculate the change in velocity $\Delta\vec{v}_s$, we must convert one of these two velocities into the other reference frame. We will convert the final velocity into the reference frame of the Earth. To do so, we will first draw a picture of the situation (shown below) in order to understand how the two frames of reference relate.



From this we see that

$$\vec{r}_{Es} = \vec{r}_{ED} + \vec{r}_{Ds}. \tag{31}$$

Taking a derivative in time yields a relationship between the velocities

$$\vec{v}_{Es} = \vec{v}_{ED} + \vec{v}_{Ds}. \tag{32}$$

Considering the final state just before the satellite impacts D2, we can identify $\vec{v}_{Es} = \vec{v}_{Esf}$ as the final velocity of the satellite in the reference frame of the Earth, $\vec{v}_{ED} = \vec{v}_{D1}$ as the velocity of D1 in the reference frame of the Earth, and $\vec{v}_{Ds} = \vec{v}_s$ as the velocity of satellite in the reference frame of D1. Making these substitutions allows us to find the final velocity of the satellite in the reference frame of the Earth to be

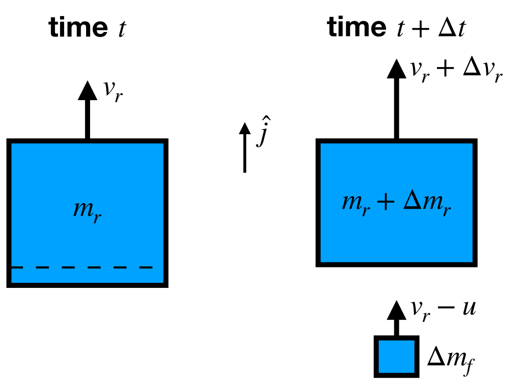
$$\vec{v}_{Esf} = \vec{v}_{D1} + \vec{v}_s. \tag{33}$$

Given that we already know that the initial velocity of the satellite in the reference frame of the Earth is $\vec{v}_{Esi} = \vec{v}_i$, we can calculate the change in velocity to be

$$\Delta\vec{v}_s = \vec{v}_{Esf} - \vec{v}_{Esi} = \vec{v}_{D1} + \vec{v}_s - \vec{v}_i. \tag{34}$$

j. This part is a continuous mass transfer problem as the satellite with its engine is behaving as a rocket. Here we will present the full derivation of the rocket equation, though it would be substantially quicker to directly start from it (or even its solution).

Since the problem statement tells us that the motion is in a straight line, we will adopt a one-dimensional coordinate system where $\hat{j}$ points in the direct of travel of the satellite. Next, at an arbitrary time t, we consider a system that is composed of the satellite including *all* the fuel it currently contains, which we will denote as having a total instantaneous mass m_r. The instantaneous speed of the satellite is v_r, so we can draw the momentum diagram shown below at time t. A very short time later at $t + \Delta t$, the satellite has ejected a differential mass element Δm_f of fuel, which slightly alters the mass of the satellite to $m_r + \Delta m_r$. Note that it is important allow an arbitrary change in the satellite's mass by $+\Delta m_r$. This will help to prevent sign errors later in the derivation (e.g. accidentally accounting for the fact that the satellite's mass is decreasing twice) and accommodate more general calculations where the mass is changing due to several mechanisms. After ejecting the differential mass element, the velocity of the satellite is also slightly changed to be $v_r + \Delta v_r$. We must also include the momentum of the ejected fuel as it is still part of the system. It has a mass of Δm_f and a velocity of $-u\hat{j}$ *relative to the satellite*. This means that it has a velocity of $(v_r - u)\hat{j}$ in the inertial laboratory frame. We have drawn the momentum diagram at time $t + \Delta t$ below.



From the momentum diagrams, we can apply conservation of mass to the system and see that

$$m_r = m_r + \Delta m_r + \Delta m_f \quad \Rightarrow \quad \Delta m_f = -\Delta m_r. \tag{35}$$

Additionally, we see that the total momentum of the system at time t is

$$\vec{p}_{sys}(t) = m_r v_r \hat{j}, \tag{36}$$

while at time $t + \Delta t$ it is

$$\vec{p}_{sys}(t+\Delta t) = (m_r+\Delta m_r)(v_r+\Delta v_r)\hat{j}+\Delta m_f(v_r-u)\hat{j} = (m_r+\Delta m_r)(v_r+\Delta v_r)\hat{j}-\Delta m_r(v_r-u)\hat{j}, \tag{37}$$

making use of equation (35). We can now write down the generalized form of Newton's second law and use the limit form of the time derivative according to

$$\vec{F}_{net}^{ext} = \frac{d\vec{p}_{sys}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{p}_{sys}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{p}_{sys}(t+\Delta t) - \vec{p}_{sys}(t)}{\Delta t}. \tag{38}$$

Since we can neglect gravitational interactions with all other astronomical bodies, we have $\vec{F}_{net}^{ext} = 0$. Using this and substituting equations (36) and (37) into the $\hat{\jmath}$ component of equation (38), we find

$$0 = \lim_{\Delta t \to 0} \frac{(m_r + \Delta m_r)(v_r + \Delta v_r) - \Delta m_r (v_r - u) - m_r v_r}{\Delta t} = \lim_{\Delta t \to 0} \frac{m_r \Delta v_r + \Delta m_r \Delta v_r + \Delta m_r u}{\Delta t}. \tag{39}$$

We can neglect the $\Delta m_r \Delta v_r$ term in this expression as it is product of two differential elements. Since the differential elements are infinitesimally small, a product of two differential elements will be much smaller than terms that include just one differential element (e.g. $\Delta m_r \Delta v_r \ll m_r \Delta v_r$). Thus, equation (39) becomes

$$0 = m_r \left(\lim_{\Delta t \to 0} \frac{\Delta v_r}{\Delta t} \right) + \left(\lim_{\Delta t \to 0} \frac{\Delta m_r}{\Delta t} \right) u. \tag{40}$$

Converting the limits back into derivatives, we find the differential equation

$$0 = m_r \frac{dv_r}{dt} + \frac{dm_r}{dt} u, \tag{41}$$

which is the standard rocket equation. To solve it, we rearrange and perform a change of variables from the time t to the mass m_r according to

$$m_r \frac{dv_r}{dt} = -\frac{dm_r}{dt} u \quad \Rightarrow \quad \frac{dv_r}{dt} \frac{dt}{dm_r} = -\frac{u}{m_r} \quad \Rightarrow \quad \frac{dv_r}{dm_r} = -\frac{u}{m_r}. \tag{42}$$

We can directly integrate this to find the satellite's speed as a function of its current mass

$$v_r(m_r) = -u \ln(m_r) + C, \tag{43}$$

where C is an integration constant. Since we know that the satellite impacts the asteroid with a mass of $m_r = m_s$, it must depart Earth with a total mass of $m_r = m_s + m_f$, where we want to calculate m_f. Fortunately, we also know the change in speed from the start to the finish of this journey

$$\Delta v_s = v_r(m_s) - v_r(m_s + m_f). \tag{44}$$

Substituting equation (43) into equation (44) allows us to eliminate the integration constant and find

$$\Delta v_s = (-u \ln(m_s) + C) - (-u \ln(m_s + m_f) + C) = u (\ln(m_s + m_f) - \ln(m_s)) = u \ln\left(\frac{m_s + m_f}{m_s}\right). \tag{45}$$

Rearranging gives the final answer of

$$\frac{m_s + m_f}{m_s} = \exp\left(\frac{\Delta v_s}{u}\right) \quad \Rightarrow \quad m_f = m_s \left(\exp\left(\frac{\Delta v_s}{u}\right) - 1 \right). \tag{46}$$

3. Tennis serve

a. This part can be solved using Newton's second law or conservation of energy. Since we are interested in properties of the ball at different positions (as opposed to different times), we guess that conservation of energy will be the quicker strategy. Since the only force acting on the ball is gravity, which is a conservative force, mechanical energy is conserved. We can write this as

$$E_{mi} = E_{mf} \quad \Rightarrow \quad \frac{m}{2}v_r^2 + mgy_r = 0 + mgy_0 \quad \Rightarrow \quad v_r = \pm\sqrt{2g\left(y_0 - y_r\right)}, \tag{1}$$

where we have ignored the rotational kinetic energy as it stays constant as the ball ascends because there are no external torques. Given our coordinate system, we know that v_r must be positive, so our final answer is

$$v_r = \sqrt{2g\left(y_0 - y_r\right)}. \tag{2}$$

b. After the ball is hit, it experiences ballistic motion as it travels towards the returner. Thus, its trajectory is parameterized by

$$x\left(t\right) = v_{x0}t + x_0 \tag{3}$$

$$y\left(t\right) = -\frac{g}{2}t^2 + v_{y0}t + y_0. \tag{4}$$

Given that the initial position is $x_0 = 0$, we can solve equation (3) for t and find

$$t = \frac{x}{v_{x0}}. \tag{5}$$

Substituting this into equation (4) gives the trajectory of the ball

$$y\left(x\right) = -\frac{g}{2v_{x0}^2}x^2 + \frac{v_{y0}}{v_{x0}}x + y_0. \tag{6}$$

c. The condition for the ball to pass above the net is

$$y\left(x_n\right) > h \tag{7}$$

as we are allowed to use $y(x)$ in our answer. Using $>$ versus $\geq$ doesn't matter as they are only different by a infinitesimal, physically-insignificant amount.

d. The condition for the ball to land before reaching the service line is

$$y\left(x_f\right) < 0 \tag{8}$$

as we are allowed to use $y(x)$ in our answer. Using $<$ versus $\leq$ doesn't matter as they are only different by a infinitesimal, physically-insignificant amount.

e. This problem is difficult. Formally, we are searching for the minimum value of y_0 that still satisfies equations (7) and (8) and also has $v_{y0} < 0$. Here we will show two methods of solving the problem: the quick way and the rigorous way. The quick way is probably the only that is possible during an exam (given the time constraints).

First, we will use a big shortcut to enable the quick way to the solution. The shortcut is to imagine the situation physically and directly deduce/guess that all three conditions must be "marginally satisfied" (i.e. you can turn the inequality symbol into an equals sign). This is possible as we have a lot of physical intuition about objects undergoing ballistic motion. It makes sense that, to hit down on the ball at the minimum possible value of y_0, you would want to hit the ball with a value of v_{y0} that was *barely* negative, then have the ball *barely* pass over the net and land *barely* before the service line. From this physical insight, we take $v_{y0} = 0$ in equation (6) and substitute it into the marginal satisfied cases of equations (7) and (8) to find

$$-\frac{g}{2v_{x0}^2}x_n^2 + y_0 = h \tag{9}$$

$$-\frac{g}{2v_{x0}^2}x_f^2 + y_0 = 0 \tag{10}$$

respectively. Rearranging equation (10) shows that

$$\frac{g}{2v_{x0}^2} = \frac{y_0}{x_f^2}, \tag{11}$$

which we can substitute into equation (9) to find

$$-\left(\frac{y_0}{x_f^2}\right)x_n^2 + y_0 = h \quad \Rightarrow \quad \left(1 - \frac{x_n^2}{x_f^2}\right)y_0 = h \quad \Rightarrow \quad y_0 = \left(1 - \left(\frac{x_n}{x_f}\right)^2\right)^{-1} h. \tag{12}$$

Now we will show an alternative method to solve this problem: the rigorous way. First, we will substitute equation (6) into equations (7) and (8) to write down

$$-\frac{g}{2v_{x0}^2}x_n^2 + \frac{v_{y0}}{v_{x0}}x_n + y_0 > h \tag{13}$$

$$-\frac{g}{2v_{x0}^2}x_f^2 + \frac{v_{y0}}{v_{x0}}x_f + y_0 < 0 \tag{14}$$

respectively. We are seeking the minimum value of y_0 for which it is possible to simultaneously satisfy equations (13) and (14) as well as

$$v_{y0} < 0. \tag{15}$$

The quantities g, x_n, x_f, and h are fixed known positive quantities, while v_{x0} and v_{y0} should be chosen to minimize y_0 while still respecting the three constraints. We can immediately argue that equation (13) should be marginally satisfied. Imagine you claimed that you had the solution for the minimum y_0 and it occurred for values of v_{x0} and v_{y0} that did *not* marginally satisfy equation (13). For any such solution, I could take it and simply reduce the value of y_0 (keeping v_{x0} and v_{y0} unchanged) until equation (13) was marginally satisfied. My solution would have a lower value of y_0 and still respect equation (13) (but just barely). It would also respect equation (14) because decreasing y_0 while maintaining v_{x0} and v_{y0} takes you further away from the limit. It would also still satisfy equation (15) as changing y_0 has no effect on v_{y0}. Thus, equation (13) must be marginally satisfied. This conclusion is intuitive. Any trajectory that travels over the net with room to spare and still lands in the green region can be shifted downwards without deformation, thereby achieving a lower y_0 while still satisfying all three constraints.

Therefore, we should rewrite equation (13) as

$$-\frac{g}{2v_{x0}^2}x_n^2 + \frac{v_{y0}}{v_{x0}}x_n + y_0 = h. \tag{16}$$

We can solve this equation for y_0 to find

$$y_0 = h + \frac{g}{2v_{x0}^2}x_n^2 + \frac{(-v_{y0})}{v_{x0}}x_n, \tag{17}$$

where we write $-v_{y0}$ in this way to remind ourselves that v_{y0} is necessarily negative according to equation (15). Substituting equation (17) into equation (14) gives

$$-\frac{g}{2v_{x0}^2}x_f^2 - \frac{(-v_{y0})}{v_{x0}}x_f + h + \frac{g}{2v_{x0}^2}x_n^2 + \frac{(-v_{y0})}{v_{x0}}x_n < 0 \quad \Rightarrow \quad h - \frac{g}{2v_{x0}^2}\left(x_f^2 - x_n^2\right) - \frac{(-v_{y0})}{v_{x0}}\left(x_f - x_n\right) < 0, \tag{18}$$

where we note that $x_f > x_n$ so both $x_f^2 - x_n^2 > 0$ and $x_f - x_n > 0$ (which is important for the coming argument). Now we will argue that this equation must be marginally satisfied. Imagine you claimed that you had the solution for the minimum y_0 and it occurred for values of v_{x0} and v_{y0} that did *not* marginally satisfy equation (18). I could take this solution and increase the value of v_{x0} (keeping v_{y0} unchanged) until equation (18) was marginally satisfied. My solution would still respect equation (18) (but just barely). It would also satisfy equation (15) as changing v_{x0} has no effect on v_{y0}. Crucially, equation (17) shows that my solution (which has a larger value of v_{x0}) would have a smaller value of y_0 than yours. Thus, equation (18) must be marginally satisfied. This result is intuitive if you imagine the situation. Due to the previous argument, here we are only considering trajectories that barely pass over the net. If such a trajectory lands before the service line with room to spare, you could hit the ball harder in the x direction. This would cause it to travel further before landing, but reach the horizontal position of the net more quickly. This reduces the vertical distance the ball falls between being hit and reaching the net, thereby enabling a lower starting position y_0.

Therefore, we should rewrite equation (18) as

$$h - \frac{g}{2v_{x0}^2}\left(x_f^2 - x_n^2\right) - \frac{(-v_{y0})}{v_{x0}}\left(x_f - x_n\right) = 0. \tag{19}$$

Solving this equation for v_{y0} yields

$$v_{y0} = \frac{v_{x0}}{x_f - x_n}\frac{g}{2v_{x0}^2}\left(x_f^2 - x_n^2\right) - \frac{v_{x0}}{x_f - x_n}h = \frac{g}{2v_{x0}}\left(x_f + x_n\right) - \frac{v_{x0}}{x_f - x_n}h. \tag{20}$$

Plugging this into equation (15) gives an lower bound on the horizontal velocity

$$\frac{g}{2v_{x0}}\left(x_f + x_n\right) - \frac{v_{x0}}{x_f - x_n}h < 0 \quad \Rightarrow \quad \frac{v_{x0}^2}{x_f - x_n}h > \frac{g}{2}\left(x_f + x_n\right) \quad \Rightarrow \quad v_{x0}^2 > \frac{g}{2h}\left(x_f^2 - x_n^2\right). \tag{21}$$

Substituting equation (20) into equation (17) gives

$$y_0 = h + \frac{g}{2v_{x0}^2}x_n^2 - \frac{x_n}{v_{x0}}\frac{g}{2v_{x0}}\left(x_f + x_n\right) + \frac{x_n}{v_{x0}}\frac{v_{x0}}{x_f - x_n}h = h - \frac{g}{2v_{x0}^2}x_n x_f + \frac{x_n}{x_f - x_n}h \tag{22}$$

$$= \frac{x_f - x_n}{x_f - x_n}h - \frac{g}{2v_{x0}^2}x_n x_f + \frac{x_n}{x_f - x_n}h = \frac{x_f}{x_f - x_n}h - \frac{g}{2v_{x0}^2}x_n x_f. \tag{23}$$

From equation (23) we see that, to minimize y_0, we want to decrease v_{x0}^2 as much as possible. Thus, equation (21) should be marginally satisfied, becoming

$$v_{x0}^2 = \frac{g}{2h}\left(x_f^2 - x_n^2\right). \tag{24}$$

Substituting this into equation (23) gives

$$y_0 = \frac{x_f}{x_f - x_n}h - \frac{g}{2}\left(\frac{2h}{g}\frac{1}{x_f^2 - x_n^2}\right)x_n x_f = \left(\frac{x_f}{x_f - x_n} - \frac{x_n x_f}{x_f^2 - x_n^2}\right)h \tag{25}$$

$$= \left(\frac{x_f\left(x_f + x_n\right)}{x_f^2 - x_n^2} - \frac{x_n x_f}{x_f^2 - x_n^2}\right)h = \frac{x_f^2}{x_f^2 - x_n^2}h = \left(1 - \left(\frac{x_n}{x_f}\right)^2\right)^{-1} h. \tag{26}$$

This is the final answer, which is consistent with the result of the first method (i.e. equation (12)) as expected.

f. Plugging the numerical values into equation (12) or (26) gives

$$y_0 = \left(1 - \left(\frac{12\text{ m}}{18\text{ m}}\right)^2\right)^{-1}(1\text{ m}) = \left(1 - \left(\frac{2}{3}\right)^2\right)^{-1}(1\text{ m}) = \left(\frac{9}{9} - \frac{4}{9}\right)^{-1}(1\text{ m}) = \frac{9}{5}\text{ m} = 1.8\text{ m}. \tag{27}$$

g. While this derivation is long, it is straightforward and follows the same method as in part b. However, in part b the ball experienced standard ballistic motion, so we could immediately write down $x(t)$ and $y(t)$. In the presence of drag, these functions are no longer obvious. Instead we must start by writing down Newton's second law, which is

$$\sum \vec{F} = m\vec{a} \quad \Rightarrow \quad -mg\hat{j} - \beta\vec{v} = m\frac{d\vec{v}}{dt} \quad \Rightarrow \quad -g\hat{j} - \frac{\beta}{m}v_x\hat{i} - \frac{\beta}{m}v_y\hat{j} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}. \tag{28}$$

First we will take the $\hat{i}$ component to find the differential equation

$$-\frac{\beta}{m}v_x = \frac{dv_x}{dt}. \tag{29}$$

This can be solved using separation of variables and the chain rule to find

$$\frac{1}{v_x}\frac{dv_x}{dt} = -\frac{\beta}{m} \quad \Rightarrow \quad \frac{d}{dt}\left(\ln\left(v_x\right)\right) = -\frac{\beta}{m} \quad \Rightarrow \quad \ln\left(v_x\right) = -\frac{\beta}{m}t + C_1 \quad \Rightarrow \quad v_x\left(t\right) = \exp\left(-\frac{\beta}{m}t\right)\exp\left(C_1\right). \tag{30}$$

Applying the initial condition for the velocity in the x direction gives

$$v_x\left(0\right) = v_{x0} = \exp\left(C_1\right) \quad \Rightarrow \quad \exp\left(C_1\right) = v_{x0}, \tag{31}$$

so equation (30) becomes

$$v_x\left(t\right) = v_{x0}\exp\left(-\frac{\beta}{m}t\right). \tag{32}$$

To find the position x we simply integrate this to find

$$x\left(t\right) = -\frac{m}{\beta}v_{x0}\exp\left(-\frac{\beta}{m}t\right) + C_2. \tag{33}$$

Applying the initial condition for the x position gives

$$x\left(0\right) = 0 = -\frac{m}{\beta}v_{x0} + C_2 \quad \Rightarrow \quad C_2 = \frac{m}{\beta}v_{x0}, \tag{34}$$

so equation (33) becomes

$$x\left(t\right) = -\frac{m}{\beta}v_{x0}\exp\left(-\frac{\beta}{m}t\right) + \frac{m}{\beta}v_{x0} = \frac{m}{\beta}v_{x0}\left(1 - \exp\left(-\frac{\beta}{m}t\right)\right). \tag{35}$$

Similarly to part b, we will need to solve this equation for t and then substitute it into $y(t)$ to find the trajectory $y(x)$. Rearranging equation (35) gives

$$\frac{\beta}{mv_{x0}}x = 1 - \exp\left(-\frac{\beta}{m}t\right), \tag{36}$$

which will be a useful intermediate step. Rearranging further gives

$$\exp\left(-\frac{\beta}{m}t\right) = 1 - \frac{\beta}{mv_{x0}}x \quad \Rightarrow \quad -\frac{\beta}{m}t = \ln\left(1 - \frac{\beta}{mv_{x0}}x\right) \quad \Rightarrow \quad t = -\frac{m}{\beta}\ln\left(1 - \frac{\beta}{mv_{x0}}x\right). \tag{37}$$

Now we will turn to the $\hat{j}$ component of equation (28)

$$-g - \frac{\beta}{m}v_y = \frac{dv_y}{dt} \quad \Rightarrow \quad \frac{dv_y}{dt} + \frac{\beta}{m}v_y = -g. \tag{38}$$

This is a first-order inhomogeneous differential equation with constant coefficients. We can solve it by writing the solution as a sum of the homogeneous solution $v_{yh}(t)$ and the inhomogeneous solution $v_{yi}(t)$

$$v_y(t) = v_{yh}(t) + v_{yi}(t). \tag{39}$$

We can guess that the inhomogeneous solution will need to be a constant to cancel the right-hand side of the differential equation (i.e. the inhomogeneous term). Substituting $v_{yi}(t) = C$ for v_y in equation (38) shows that

$$\frac{dv_{yi}}{dt} + \frac{\beta}{m}v_{yi} = -g \quad \Rightarrow \quad 0 + \frac{\beta}{m}C = -g \quad \Rightarrow \quad C = -\frac{mg}{\beta} \quad \Rightarrow \quad v_{yi}(t) = -\frac{mg}{\beta}. \tag{40}$$

To find the homogeneous solution we substitute the whole form of the solution (i.e. equation (39) along with equation (40)) into the differential equation of equation (38) to see

$$\frac{dv_{yh}}{dt} + \frac{dv_{yi}}{dt} + \frac{\beta}{m}v_{yh} + \frac{\beta}{m}v_{yi} = -g \quad \Rightarrow \quad \frac{dv_{yh}}{dt} + 0 + \frac{\beta}{m}v_{yh} - g = -g \quad \Rightarrow \quad -\frac{\beta}{m}v_{yh} = \frac{dv_{yh}}{dt}. \tag{41}$$

This differential equation has an identical form to equation (29), so we can look at the solution of equation (30) and write down

$$v_{yh}\left(t\right) = \exp\left(-\frac{\beta}{m}t\right)\exp\left(C_3\right). \tag{42}$$

By substituting this and equation (40) into equation (39), we find the full solution of

$$v_y(t) = \exp\left(-\frac{\beta}{m}t\right)\exp\left(C_3\right) - \frac{mg}{\beta}. \tag{43}$$

Applying the initial condition for the velocity in the y direction gives

$$v_y\left(0\right) = v_{y0} = \exp\left(C_3\right) - \frac{mg}{\beta} \quad \Rightarrow \quad \exp\left(C_3\right) = v_{y0} + \frac{mg}{\beta}, \tag{44}$$

so equation (43) becomes

$$v_y(t) = \left(v_{y0} + \frac{mg}{\beta}\right)\exp\left(-\frac{\beta}{m}t\right) - \frac{mg}{\beta}. \tag{45}$$

To find the position y we simply integrate this to get

$$y\left(t\right) = -\frac{m}{\beta}\left(v_{y0} + \frac{mg}{\beta}\right)\exp\left(-\frac{\beta}{m}t\right) - \frac{mg}{\beta}t + C_4. \tag{46}$$

Applying the initial condition for the y position gives

$$y\left(0\right) = y_0 = -\frac{m}{\beta}\left(v_{y0} + \frac{mg}{\beta}\right) + C_4 \quad \Rightarrow \quad C_4 = \frac{m}{\beta}\left(v_{y0} + \frac{mg}{\beta}\right) + y_0, \tag{47}$$

so equation (46) becomes

$$y\left(t\right) = -\frac{m}{\beta}\left(v_{y0} + \frac{mg}{\beta}\right)\exp\left(-\frac{\beta}{m}t\right) - \frac{mg}{\beta}t + \frac{m}{\beta}\left(v_{y0} + \frac{mg}{\beta}\right) + y_0 \tag{48}$$

$$= \frac{m}{\beta}\left(v_{y0} + \frac{mg}{\beta}\right)\left(1 - \exp\left(-\frac{\beta}{m}t\right)\right) - \frac{mg}{\beta}t + y_0. \tag{49}$$

Similarly to part b, we will now substitute in our solution for t from the x component of the motion. However, we can save some effort by substituting equation (36) as well as equation (37) (where appropriate). This yields the final answer of

$$y\left(x\right) = \frac{m}{\beta}\left(v_{y0} + \frac{mg}{\beta}\right)\left(\frac{\beta}{mv_{x0}}x\right) - \frac{mg}{\beta}\left(-\frac{m}{\beta}\ln\left(1 - \frac{\beta}{mv_{x0}}x\right)\right) + y_0 \tag{50}$$

$$= \frac{1}{v_{x0}}\left(v_{y0} + \frac{mg}{\beta}\right)x + \frac{m^2}{\beta^2}g\ln\left(1 - \frac{\beta}{mv_{x0}}x\right) + y_0. \tag{51}$$