Name (Last, First):

Instructions

1. Write clearly and in ink (pen), and indicate clearly your result (with your reasoning explained).
2. Only responses written on the A4 sheets of the exam (front and back) will be taken into account.
3. Justify your calculations and responses with diagrams and by describing the laws of physics you have used.
4. This exam should be taken in 3 hours.

Allowed:

- one page A4 (front and back) *handwritten by you*
- language dictionary

Not allowed:

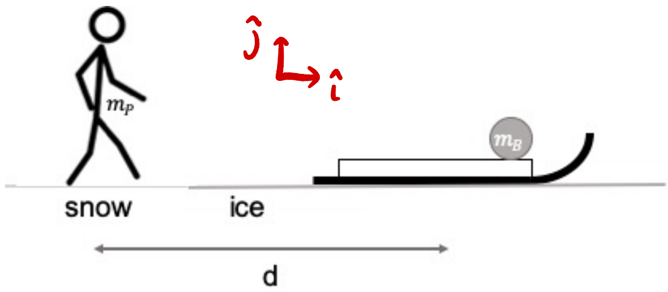
- Any form of electronic device including a calculator
- Leaving the room without authorization

(Do not mark on this page below this line)

NO CONTENT ON THIS PAGE

1. Snowball throw (10 pts total)

A person of mass m_P loads their sled with a snowball, of mass m_B. The person then pushes the sled onto a frictionless, horizontal icy lake, leaving it at rest a distance d away, as shown in the diagram. The mass of the person plus the empty sled together is M_S. Ignore air resistance for this exercise. **<u>For (a) and (b), draw the relevant diagram(s).</u>**



4

a) The person jumps with initial velocity $\vec{v}_P$. What are the conditions on the vector components of $v_{P,x}$ and $v_{P,y}$ so that the person lands in the sled, then the person+sled slides with speed v_0? Write your answers in terms of g (the gravitational acceleration at the earth's surface), d, m_P, m_B, M_S and v_0. (Hint: Do not use the range equation)

Two stages of motion. ① jump (projectile motion) ② inelastic collision.

② To end up with a final speed v_0, consider the system sled + person. Only need to consider $\hat{i}$ for this part, since sled moves in 1D.

initial: $m_P \rightarrow v_{p,x}$; m_b, $M_s - m_p$

All forces are conservative ⇒ momentum is conserved.

final: m_p, m_b, $M_s - m_p$ $\rightarrow v_0$

$$m_p v_{p,x} + (M_s - m_p + m_b)\cdot 0 = (M_s + m_b) v_0$$

$$\Rightarrow \boxed{v_{p,x} = \frac{M_s + m_b}{m_p} v_0}$$

1 pt. momentum diagram

1 pt. apply conservation of momentum

① The person must remain in the air long enough to travel a horizontal distance d.

$$t_{jump} = \frac{d}{v_{p,x}} = \frac{d\, m_p}{(M_s + m_b) v_0}$$

$$v_{p,y} = g t_{fall} \quad \text{vertical velocity of an object after time } t_{fall}\text{, starting from rest}$$

$$= \frac{1}{2} g t_{jump}$$

$$\boxed{= \frac{1}{2} g \frac{d\, m_p}{(M_s + m_b) v_0}}$$

1 pt. motion under const. velocity to horizontal

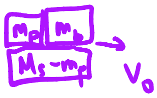
1 pt. motion under const. acceleration to vertical

$\boxed{4}$

1. Snowball throw

b) The person and sled are sliding with speed v_0. The person throws the snowball horizontally off the back of the sled. The snowball has an initial horizontal velocity v_b with respect to the ice, in the same direction as the motion of the sled. If instead the person throws the snowball horizontally off the front of the sled, in the direction of its motion, what would be the speed v_f of the snowball with respect to the ice? Assume the person throws with the same strength, and thus gives the snowball the same speed relative themselves in each case. Write your answer in terms of only v_0 and v_b.

initial:



final:

$V_b = \frac{1}{2} V_0$

1 pt momentum diagram

1 pt. apply conservation of momentum

$$\text{Conservation of momentum: } \underbrace{(m_b + M_s) v_0}_{\text{initial}} = \underbrace{m_b v_b + M_s U_b}_{\text{final, back}} \Rightarrow U_b = \frac{1}{M_s}\left[(m_b + M_s) v_0 - m_b v_b\right]$$

Quantities in the reference frame of the sled + person are denoted $'$. ΔV is the relative speed between ball & person.

$$\text{transform: } v_b' = v_b - U_b = -\Delta V = v_b - \frac{m_b}{M_s} v_0 - v_0 + \frac{m_b}{M_s} v_b = (v_b - v_0)\left(\frac{M_s + m_b}{M_s}\right)$$

Constraint: ΔV is the same for both throws, since thrown "with same strength"

Throw from front

$$\text{Conservation of momentum: } \underbrace{(m_b + M_s) v_0}_{\text{initial}} = \underbrace{m_b v_f + M_s U_f}_{\text{final, front}}$$

$$\Rightarrow U_f = \frac{1}{M_s}\left[(m_b + M_s) v_0 - m_b v_f\right]$$

1 pt. transform between reference frames

$$\text{transform: } v_f' = v_f - U_f = \Delta V \Rightarrow v_f = \Delta V + \frac{m_b + M_s}{M_s} v_0 - \frac{m_b}{M_s} v_f \Rightarrow v_f\left(\frac{M_s + m_b}{M_s}\right) = \Delta V + \frac{m_b + M_s}{M_s} v_0$$

$$v_f = \left(\frac{M_s}{M_s + m_b}\right)\Delta V + v_0 = v_0 + (v_0 - v_b)\left(\frac{M_s + m_b}{M_s}\right)\left(\frac{M_s}{M_s + m_b}\right) \boxed{= 2v_0 - v_b}$$

1 pt. express in terms of given variables

$\boxed{2}$

c) After throwing the snowball, the sled+person has a velocity u_f. The sled runs off the lake onto the snow, where the coefficient of kinetic friction is μ_k. What is the distance L the sled will slide before coming to a stop, in terms of in terms of g, μ_k, and u_f?

Work-kinetic energy theorem: $\Delta K = W$

$$0 - \frac{1}{2} M_s U_f^2 = \int_0^L \vec{F} \cdot d\vec{x} = -\mu_k N x \Big|_0^L = -\mu_k M_s g L$$

$$\Rightarrow \boxed{L = \frac{1}{2} \frac{U_f^2}{\mu_k g}}$$

1 pt. work-kinetic energy theorem

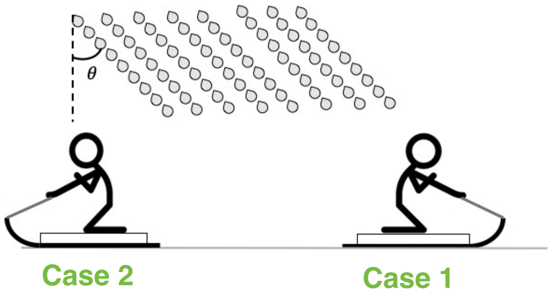
1 pt. solve for L

OR: 1 pt. motion const. acceleration

1 pt. solve for L

2. Sleeting sledding (10 pts total)

An icy rain begins to fall. An observer in the ground frame measures that it is falling with a speed v_R at an angle θ with respect to the vertical. **For (a) and (c), draw the relevant diagram(s).**



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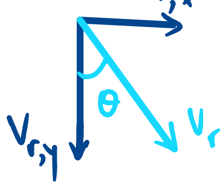
a) To the person in the sled described in Exercise 1 sliding with speed u_f, the rain appears to fall vertically (Case 1). If the person turns the sled to slide with the same speed but in the opposite direction (Case 2), the rain appears to fall at an angle φ. What is θ in terms of φ?

Ground frame

$V_{r,x} = V_r \sin\theta$



$V_{r,y} = V_r \cos\theta$

1 pt triangles

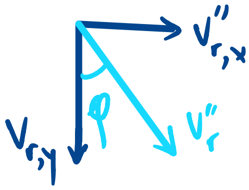
Case 1 sled frame

$V'_{r,x} = V_{r,x} - U_f = 0$

$V'_{r,y} = V_{r,y}$

$\Rightarrow V_{r,x} = U_f = V_r \sin\theta$

Case 2 sled frame



$V''_{r,x} = V_{r,x} + U_f$

1 pt. reference frames

$\Rightarrow V''_{r,x} = 2V_{r,x} = 2V_r \sin\theta$

$V''_{r,y} = V_{r,y} = V_r \cos\theta$

$\Rightarrow \tan\varphi = 2\tan\theta$

1 pt solve for $\theta(\varphi)$

or $\boxed{\theta = \tan^{-1}\left(\frac{1}{2}\tan\varphi\right)}$

For (b) and (c), suppose the rain falls at a constant rate σ/τ (kg /m²/s), and accumulates in the sled whose area is A.

1

b) Before the rain begins, the mass of the person+sled system is M_S. What is the mass of the system as a function of time, $M(t)$?

Increase in mass: $\frac{\sigma A}{\tau} \cdot t$

1 pt

$$\Rightarrow \boxed{M(t) = M_s + \frac{\sigma A t}{\tau}}$$

6

2. Sleeting sledding

c) Use the concept of momentum transfer to find the differential equation for the derivative of the velocity of the sled as a function of time in Case 1 and in Case 2, dV_{S1}/dt and dV_{S2}/dt. **Do not** solve the differential equations by integrating. Write your answers in terms of some or all of v_R, θ, V_{S1}, V_{S2}, σ, τ, M_S and A. (Hint: Terms that are a product of two differential elements are small, and can be set equal to zero.)

Case 1 : the rain has the same horizontal velocity as the sled.

So, it doesn't change the momentum of the sled, only the mass.

$$\boxed{dV_{s1}/dt = 0}$$

1 pt reasoning

1 pt. doesn't change

Case 2 : momentum is transferred to the sled.

time t: $\boxed{\Delta m} \rightarrow v_r \sin\theta$, $\boxed{M} \rightarrow V_{s,2}$

time $t+\Delta t$: $\boxed{M+\Delta m} \rightarrow V_{s,2} + \Delta V_{s,2}$

$\vec{\imath}$

1 pt momentum diagram

$$\vec{F}_{ext} = 0 \Rightarrow \vec{p}_{sys}(t+\Delta t) = \vec{p}_{sys}(t)$$

$$(M+\Delta m)(V_{s,2}+\Delta V_{s,2}) = (M V_{s,2} + \Delta m\, v_r \sin\theta)$$

$$\Rightarrow M\Delta V_{s,2} + \underbrace{\Delta m \Delta V_{s,2}}_{\text{negligible}} = \Delta m\,(v_r \sin\theta - V_{s,2})$$

1 pt momentum principle

$\Delta m = \Delta M$ since sled's mass changes from rain. Take lim as $\Delta t \to 0$.

$$M\,dV_{s2} = dM\,(v_r \sin\theta - V_{s2}) \Rightarrow \frac{dV_{s2}}{dt} = \frac{1}{M}\frac{dM}{dt}(v_r \sin\theta - v_{s2})$$

1 pt take limit

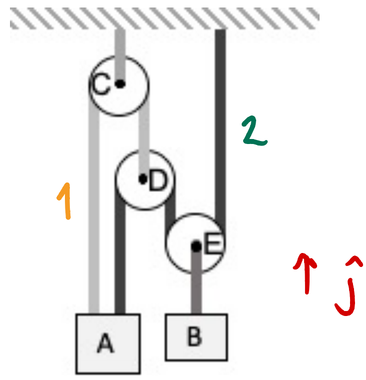
Use $M(t)$ from (b).

$$\boxed{\frac{dV_{s2}}{dt} = \frac{\sigma A\,(v_r \sin\theta - V_{s,2})}{\tau M_s + \sigma A t}}$$

1 pt final expression

✓ If $v_r \sin\theta = V_{s,2}$ rain appears vertical and acceleration $\rightarrow 0$

3. Pulling pulleys (10 pts total)



Two blocks, A and B, with masses m_A and m_B can move along the vertical axis, connected by a system of ropes and pulleys (C, D, E). Assume the pulleys are massless and frictionless, and the ropes are massless and inextensible. Rope 1 connects block A to pulley D (light grey), and rope 2 connects block A to the ceiling (black).

3

a) Derive the constraint equation that relates the vertical component of the acceleration of block A, a_A, to that of block B, a_B.

1 strategy: virtual displacement or differential of total length

Virtual displacement argument:

If A moves up by Δx_A, slack in 1, 3 of Δx_A each

Slack in 1 is transmitted to D ⇒ Slack in 2 of $2\Delta x_A$

Total slack in 2 is $3\Delta x_A$

Divided evenly between rope on each side of E. (Divide by 2)

1 analysis

$$\Rightarrow m_B \text{ moves by } \Delta x_B = -\frac{3}{2}\Delta x_A$$

$$\Rightarrow a_B = \frac{-3}{2} a_A \quad \text{or} \quad \boxed{a_A = -\frac{2}{3} a_B}$$

1 $a_A(a_B)$

1

b) At a time t^*, the velocity of block B is v_B^*, downward. What are the velocities of block A and pulley D, v_A^* and v_D^*?

$$\boxed{V_A^* = -\frac{2}{3} V_B^*}$$

1

$$\boxed{V_D^* = -V_A^* = \frac{2}{3} V_B^*}$$

6

3. Pulling pulleys

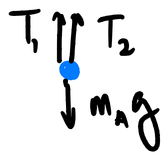
c) Find the tension in rope 1 and rope 2, T_1 and T_2, as a function of m_A, m_B, and g, the gravitational acceleration at the earth's surface. Under what condition will the system be in static equilibrium? Check that your answer makes sense when compared with part (a).

Note: Tension in a massless rope is constant. $\boxed{1 \; T_1 \neq T_2}$

$\boxed{1 \text{ Newton's 2nd}}$

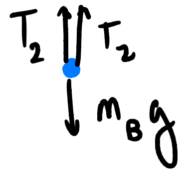
A

$\boxed{1 \text{ FBD}}$



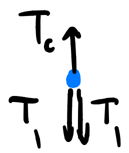
$$T_1 + T_2 - m_A g = m_A a_A \Rightarrow a_A = \frac{1}{m_A}\left[T_1 + T_2 - m_A g\right]$$
$$= \frac{1}{m_A}\left[3T_2 - m_A g\right]$$

B+E



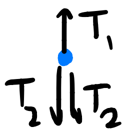
$$2T_2 - m_B g = m_B a_B \qquad a_B = \frac{1}{m_B}\left[2T_2 - m_B g\right]$$

C



$$T_c - 2T_1 = 0$$

D



$$T_1 - 2T_2 = 0 \Rightarrow T_1 = 2T_2$$

From (a) : $a_A = -\frac{2}{3} a_B$

$$\frac{1}{m_A}\left[3T_2 - m_A g\right] = \frac{-2}{3m_B}\left[2T_2 - m_B g\right]$$

$$T_2\left[\frac{3}{m_A} + \frac{4}{3m_B}\right] = g + \frac{2}{3}g = \frac{5}{3}g$$

$\boxed{1 \; T_2}$

$$T_2 = \frac{5}{3}g\left[\frac{3}{m_A} + \frac{4}{3m_B}\right]^{-1} = \frac{5}{3}g\left[\frac{9m_B + 4m_A}{3m_A m_B}\right]^{-1} = \boxed{\frac{5g\, m_A m_B}{9m_B + 4m_A}}$$

$$\boxed{T_1 = \frac{10g\, m_A m_B}{9m_B + 4m_A}}$$

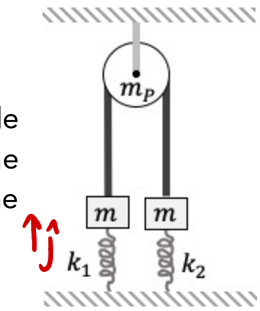
$\boxed{1 \; T_1}$

In static equilibrium, $a_A = 0$, $a_B = 0 \Rightarrow T_2 = \frac{m_A g}{3} = \frac{m_B g}{2}$ $\boxed{1 \text{ check}}$

So, $\boxed{m_A = \frac{3}{2} m_B}$ allows force balance in (a)

4. Spring fling (10 pts total)

Two blocks of equal mass m are attached by a massless, inextensible string. The string wraps over a massless pulley, which hangs from the ceiling. Each mass is attached to a spring anchored to the floor. The spring constants are k_1 and k_2 as shown in the diagram.



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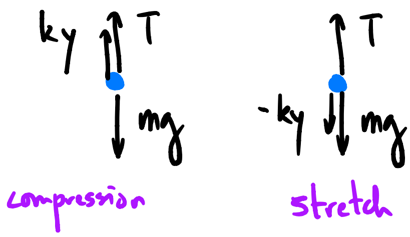
a) Initially, the system is at rest, and both springs are at their equilibrium lengths. The block on the left is pulled down by a distance y_0 and released from rest. Assume $k_1 < k_2$. What is the acceleration of each block, $a_{1,i}$ (left block), and $a_{2,i}$ (right block) immediately after the release? Specify both the magnitude and direction.

Newton's 2nd Law:



1 FBD

For any position Δy from equilibrium:

$$\Sigma F : \begin{cases} T + k_1 y_1 - mg = m a_1 \\ T - k_2 y_2 - mg = m a_2 \end{cases}$$

Choose origin at initial positions.

1 Newton's 2nd law

Constraint: $a_1 = -a_2$ as long as $T > 0$

1 constraint

Compression of 1 by y_0 causes stretching of 2 by y_0

$$\begin{cases} T + k_1 y_0 - mg = m a_i \quad (1) \\ T - k_2 y_0 - mg = -m a_i \quad (2) \end{cases}$$

1 magnitude

$(1) - (2)$: $k_1 y_0 + k_2 y_0 = 2 m a_i \Rightarrow \boxed{a_i = \frac{(k_1 + k_2)}{2m} y_0}$

1 direction

$$\boxed{\vec{a}_{1i} = a_i \hat{j} \qquad \vec{a}_{2i} = -a_i \hat{j}}$$

3

4. Spring fling

b) What is the tension in the string, T? Under what constraint on the spring constants would the accelerations of the two blocks be independent of each other? In this case, what values would $a_{1,i}$ and $a_{2,i}$ take on?

(1) + (2) : $2T + (k_1 - k_2) y_0 - 2mg = 0$

$$T = mg - (k_1 - k_2)\frac{y_0}{2}$$

1 tension

A negative tension is non-physical, and at $T=0$ the motion becomes uncoupled.

$$(k_1 - k_2)\frac{y_0}{2} > mg \Rightarrow \boxed{k_1 - k_2 > \frac{2mg}{y_0}}$$

1 condition

$$\boxed{a_{1i} = \frac{k_1 y_0}{m} - g} \quad \text{and} \quad \boxed{a_{2i} = \frac{k_2 y_0}{m} + g}$$

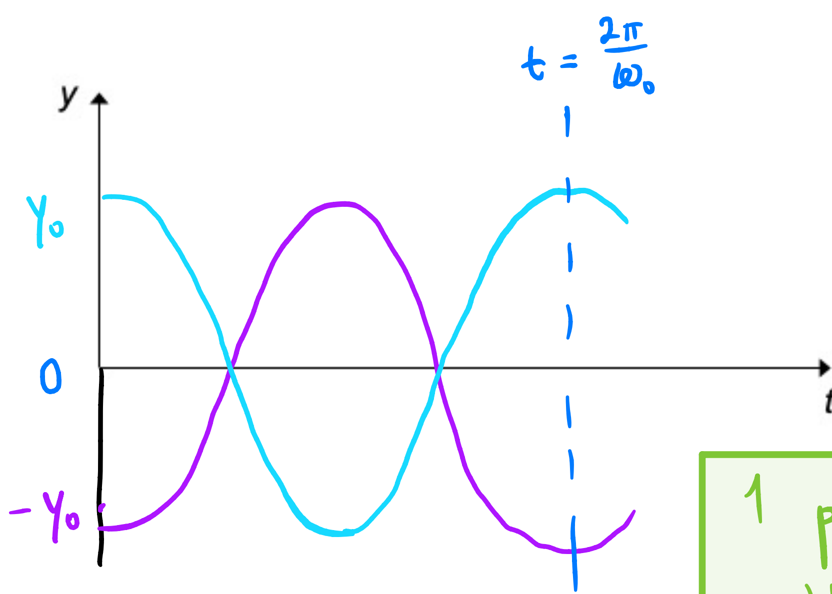
1 values

2

c) What natural oscillatory frequency, ω_0, does the system take on in case (b)? Graph the position of both blocks with the axes provided below, be sure to clearly distinguish the motion of each block, and to note key features or values particular to the motion.

$$a = \ddot{y} = -\frac{(k_1 + k_2)}{2m} y \Rightarrow \boxed{\omega_0 = \sqrt{\frac{(k_1 + k_2)}{2m}}}$$

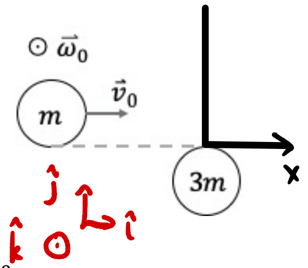
1 ω_0



1 plot, π out of phase, amplitude y_0, period $\frac{2\pi}{\omega_0}$

5. Sticky disks (10 pts total)

A uniform disk (disk 1) of mass m and radius R slides on a frictionless table with center of mass velocity $\vec{v}_0$, while spinning with an angular velocity $\vec{\omega}_0$ about its center of mass. It contacts a second disk (disk 2) of mass $3m$ and radius R which is at rest, and instantly sticks, so that the two move as a single, rigid object. Note: the moment of inertia of disk 1 about an axis through its center of mass (out of the page as drawn) is $\frac{1}{2}mR^2$. Write your answers in terms of the given variables m, R, $\vec{v}_0$, and $\vec{\omega}_0$.



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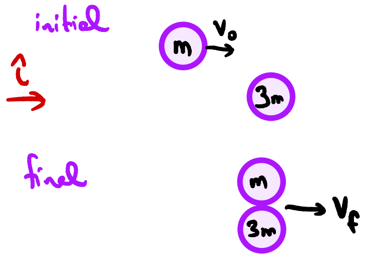
a) What is the vertical position of the center of mass of the two-disk system, y_{cm}? What are the translational and rotational velocities about the system's center of mass after the collision, $\vec{v}_f$ and $\vec{\omega}_f$? Indicate both magnitudes and directions.

$$y_{cm} = \frac{Rm - R(3m)}{4m} = -\frac{1}{2}R$$

1 center of mass

No external forces ⇒ linear and angular momenta are conserved.

Linear:



in $\hat{i}$

$p_i = mv_0$

$p_f = 4mv_f$

$p_i = p_f$

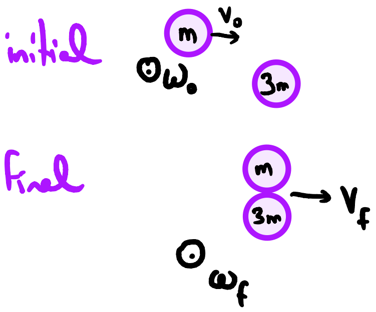
$mv_0 = 4mv_f$

$$\Rightarrow \boxed{\vec{v}_f = \frac{1}{4}v_0\,\hat{i}}$$

1 conservation of momentum

1 magnitude, direction

Angular: Want to find rotational velocity about center of mass.



Steiner: $I_f = \frac{1}{2}mR^2 + m\,d_1^2 + \frac{1}{2}(3m)R^2 + 3m\,d_2^2$

$$= \frac{1}{2}mR^2 + m\underbrace{\left(\frac{3}{2}R\right)^2}_{9/4} + \frac{3}{2}mR^2 + 3m\underbrace{\left(\frac{1}{2}R\right)^2}_{3/4} = 5mR^2$$

1 moment of inertia + Steiner

$$\vec{L}_i = \frac{1}{2}mR^2\omega_0\hat{k} - \frac{3}{2}mRv_0\hat{k} = I_f\vec{\omega}_f = \vec{L}_f$$

$$\boxed{\omega_f = \left(\frac{1}{10}\omega_0 - \frac{3}{10}\frac{v_0}{R}\right)\hat{k}}$$

1 magnitude, direction

1

5. Sticky disks

b) For what value of $\vec{\omega}_0$ would the system after the collision not rotate?

The two terms must cancel to set $\omega_f = 0$

1

$$\boxed{\omega_0 = 3\frac{v_0}{R}}$$

4

c) How much mechanical energy, ΔE, is lost in the collision, under the condition that the system does not rotate after the collision? Where did this mechanical energy likely go?

Total mechanical energy is purely kinetic.

$$E_i = K_i = \frac{1}{2}mv_0^2 + \frac{1}{2}I_1\omega_0^2$$

1 trans + rot kinetic energy

$$= \frac{1}{2}mv_0^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(3\frac{v_0}{R}\right)^2$$

$$= \frac{1}{2}mv_0^2 + \frac{3}{4}mv_0^2 = \frac{5}{4}mv_0^2$$

$$E_f = K_f = \frac{1}{2}(4m)v_f^2 = 2m\left(\frac{1}{4}v_0\right)^2$$

1 trans kinetic energy

$$= \frac{1}{8}mv_0^2$$

$$\Delta E = \left(\frac{1}{8} - \frac{5}{4}\right)mv_0^2 = \boxed{-\frac{9}{8}mv_0^2}$$

1 ΔE

Mechanical energy was converted into another form of energy, probably thermal.

1 explanation