Solutions of Exercises of Chapter 8

Solution 8.1: The graphical representation of $y(kh)$ is represented in figure 1, which gives :

$$y(kh) = \{\mathbf{0}, 0.5, 1.5, 3, 3, 2.5, 1.5, 0, 0, 0, ...\}$$

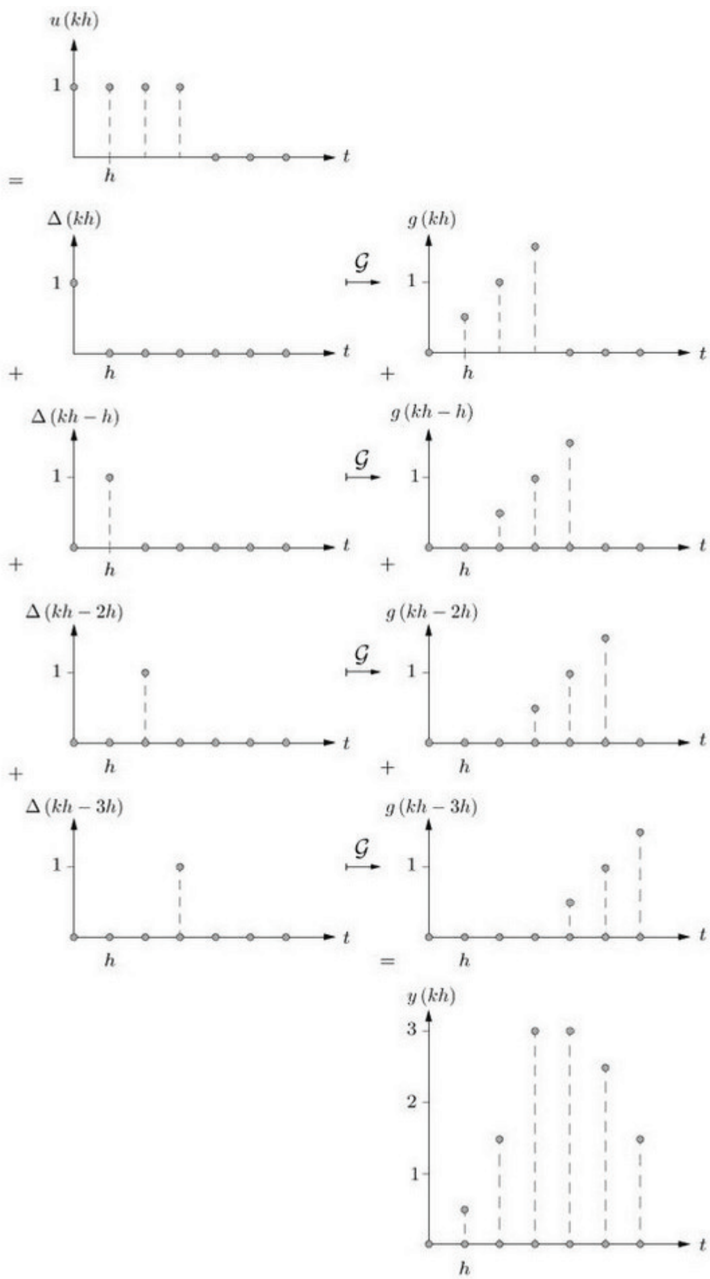

Figure 1: Graphical evaluation of the output of a discrete system.

Solution 8.2:

1. The sampling frequency should be chosen between 20 to 50 times the closed-loop bandwidth. Therefore, it can be chosen between 200 to 500 rad/s. Let's choose it as $\omega_s = 400$ rad/s then the sampling time will be $h = 2\pi/\omega_s = 15.7ms$.

   The cutoff frequency of the anti-aliasing filter should be larger than the bandwidth of the system and smaller than the Nyquist frequency (half of the sampling frequency), i.e. $10 < \omega_c < 200$. A forth-order butterworth filter with a cutoff frequency of $\omega_c = 100$ rad/s is given by:

$$F(s) = \frac{1}{\left(\frac{s}{\omega_c}\right)^4 + 2.613\left(\frac{s}{\omega_c}\right)^3 + 3.411\left(\frac{s}{\omega_c}\right)^2 + 2.613\left(\frac{s}{\omega_c}\right) + 1}$$

2. In order to have between 5 and 10 samples during the rise-time the sampling period can be chosen $2 < h < 4$ ms. If we choose $h = 3$ ms then the sampling frequency will be $\omega_s = 2 * \pi/h = 2094$ rad/s. The cutoff frequency of the anti-aliasing filter can be chosen as $\omega_c = 1000$ rad/s and be applied by the above butter-worth filter.

Solution 8.3:

(a) We note that the signal $w(kh)$ is equal to :

$$\begin{aligned}\{...,0,\mathbf{0},0,0,2,2,2,2,2,0,0,0,0,...\} &= 2\{...,0,\mathbf{0},0,0,1,1,1,1,1,1,1,...\}\\ &\quad - 2\{...,0,\mathbf{0},0,0,0,0,0,0,0,1,1,...\}\end{aligned}$$

which is the difference between two delayed discrete step signals. The first step signal has a delay of 3 and the second step has a delay of 8. So the Z transform will be:

$$W(z) = 2z^{-3}\frac{z}{z-1} - 2z^{-8}\frac{z}{z-1} = 2\frac{z^5-1}{z^8-z^7}$$

where $z/(z-1)$ is the Z transform of a step signal. The polynomial division of the numerator of this rational function by it's denominator gives :

$$W(z) = 2(z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7})$$

which has a convergence radius of $r = 0$. Note that the same result could be obtained directly by applying the definition of the Z transform.

(b) We know that for a discrete ramp signal we have

$$\mathcal{Z}\{kh\} = \frac{hz}{(z-1)^2} \qquad |z| > 1 = r$$

Using the complex derivative property of the Z transform, we have:

$$\begin{aligned}\mathcal{Z}\{\frac{1}{2}(kh)^2\} &= \frac{1}{2}\mathcal{Z}\{khkh\} = -\frac{1}{2}hz\frac{d}{dz}\left(\frac{hz}{(z-1)^2}\right)\\ &= -\frac{1}{2}hz\frac{h(z-1)^2 - 2(z-1)hz}{(z-1)^4} = \frac{h^2 z(z+1)}{2(z-1)^3} \qquad |z| > 1 = r\end{aligned}$$

(c) The partial fraction decomposition is of the form:

$$W(z) = c_0 + \frac{c_1 z}{z-2} + \frac{c_2 z}{z+2} + \frac{c_3 z}{z+3}$$

Calculation of the coefficients c_0, c_1, c_2 and c_3 :

$$\begin{aligned}
c_0 &= -\frac{2}{3} \\
c_1 &= \lim_{z\to 2}\left(\frac{z-2}{z}\cdot\frac{z^2-3z+8}{(z-2)(z+2)(z+3)}\right) = \frac{3}{20} \\
c_2 &= \lim_{z\to 2}\left(\frac{z+2}{z}\cdot\frac{z^2-3z+8}{(z-2)(z+2)(z+3)}\right) = \frac{9}{4} \\
c_3 &= \lim_{z\to 2}\left(\frac{z+3}{z}\cdot\frac{z^2-3z+8}{(z-2)(z+2)(z+3)}\right) = -\frac{26}{15}
\end{aligned}$$

Therefore :

$$w(kh) = -\frac{2}{3}\Delta(kh) + \frac{3}{20}2^k + \frac{9}{4}(-2)^k - \frac{26}{15}(-3)^k \qquad k \geq 0$$

(d) The numerical inversion formula gives, with $a_1 = -3, a_2 = 2, b_0 = 0, b_1 = 1$ and $b_2 = 3$:

$$\begin{aligned}
w(0) &= 0 \\
w(h) &= 1 - 0(-3) \\
w(2h) &= 3 - (0 \times 2 + 1(-3)) = 6 \\
w(3h) &= 0 - (0 \times 0 + 1 \times 2 + 6(-3)) = 16 \\
w(4h) &= 0 - (0 \times 0 + 1 \times 0 + 6 \times 2 + 16(-3)) = 36
\end{aligned}$$

We deduce:

$$\{w(kh)\} = \{\ldots, \mathbf{0}, 1, 6, 16, 36, \ldots\}$$

(e) It is clear that for $a > 0$ the exponential term e^{-akh} goes to zero when $k \to \infty$. Therefore, as $\sin(\omega kh)$ is bounded the signal converges to zero. For $a \leq 0$ there will be no limit for the signal when $k \to \infty$. This can be shown by the final value theorem as well. The Z transform of $w(kh) = e^{-akh}\sin(\omega kh)$ is:

$$W(z) = \frac{e^{-ah}\sin(\omega h)z}{z^2 + 2e^{-ah}\cos(\omega h)z + e^{-2ah}}$$

The poles of $W(z)$ are:

$$\begin{aligned}
e^{-ah}\cos(\omega h) \pm \sqrt{e^{-2ah}\cos^2(\omega h) - e^{-2ah}} &= e^{-ah}\left(\cos(\omega h) \pm \sqrt{\cos^2(\omega h) - 1}\right) \\
&= e^{-ah}\left(\cos(\omega h) \pm j\sin(\omega h)\right) = e^{-ah}e^{\pm j\omega h}
\end{aligned}$$

The modulus e^{-ah} of these poles is strictly smaller than 1 if and only if $a > 0$. In this case, the final value theorem can be employed to give :

$$\lim_{k\to\infty} \omega(kh) = \lim_{z\to 1}(z-1)\frac{e^{-ah}\sin(\omega h)z}{z^2 - 2e^{-ah}\cos(\omega h)z + e^{-2ah}} = 0$$

When $a \leq 0$, one hypothesis of the final value theorem is not satisfied. A wrong result, $\lim_{k\to\infty} w(kh) = 0$, is obtained if this theorem is used, even though this limit does not exist.

Solution 8.4: The Z transform of a discrete unit step signal is:

$$U(z) = \frac{z}{z-1}$$

The step response is:

$$Y(z) = G(z)U(z) = \frac{0.393}{z-0.607}\frac{z}{z-1} = c_0 + \frac{c_1 z}{z-0.607} + \frac{c_2 z}{z-1}$$

The calculation of the coefficients c_0, c_1 and c_2 gives:

$$\begin{aligned}
c_0 &= 0 \\
c_1 &= \lim_{z\to 0.607}\left(\frac{z-0.607}{z}\frac{0.393z}{(z-0.607)(z-1)}\right) = -1 \\
c_2 &= \lim_{z\to 1}\left(\frac{z-1}{z}\frac{0.393z}{(z-0.607)(z-1)}\right) = 1
\end{aligned}$$

Then

$$y(kh) = -(0.607)^k + 1 \quad , \quad k \geq 0$$

Solution 8.5: The ZOH conversion formula leads to:

$$H(z) = (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{\gamma}{s^3}\right)\right\}$$

The Z transform table, gives:

$$\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1}{s^3}\right)\right\} = \frac{h^2 z(z+1)}{2(z-1)^3}$$

Therefore,

$$H(z) = \gamma\frac{z-1}{z}\frac{h^2 z(z+1)}{2(z-1)^3} = \gamma\frac{h^2}{2}\frac{z+1}{(z-1)^2}$$

Solution 8.6: The transfer function $H(z) = Y(z)/U(z)$ is computed as follows:

$$\begin{aligned}
H(z) &= (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1}{s(s+1)}\right)\right\} = \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+1}\right)\right\} \\
&= \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-h}}\right) = 1 - \frac{z-1}{z-e^{-h}} \\
&= \frac{1-e^{-h}}{z-e^{-h}}
\end{aligned}$$

Introducing $U(z) = \dfrac{z}{z-1}$ we have:

$$Y(z) = H(z)U(z) = \frac{1-e^{-h}}{z-e^{-h}}\frac{z}{z-1} = c_0 + \frac{c_1 z}{z-e^{-h}} + \frac{c_2 z}{z-1}$$

where:

$$\begin{aligned}
c_0 &= Y(0) = 0 \\
c_1 &= \lim_{z\to e^{-h}}\left(\frac{z-e^{-h}}{z}\frac{(1-e^{-h})z}{(z-e^{-h})(z-1)}\right) = -1 \\
c_2 &= \lim_{z\to 1}\left(\frac{z-1}{z}\frac{(1-e^{-h})z}{(z-e^{-h})(z-1)}\right) = 1
\end{aligned}$$

Therefore:

$$y(kh) = \mathcal{Z}^{-1}(Y(z)) = \mathcal{Z}^{-1}\left(-\frac{z}{z-e^{-h}} + \frac{z}{z-1}\right) = 1 - e^{-kh},\ \ k \geq h$$

The step response of the isolated continuous-time model is:

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\frac{1}{s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+1}\right) = 1 - e^{-t},\ \ t \geq 0$$

Sampling this signal with the sampling period h, we obtain:

$$(1-e^{-t})\big|_{t=kh} = 1 - e^{-kh},\ \ k \geq 0$$

This result is identical to the step response of the system given in the figure because the D-A converter transforms the discrete unit step into, exactly, an analog unit step. This equivalence between signals is not valid for any input $u(kh)$. For instance, a discrete ramp $u(kh) = kh, k \geq 0$ does not yield at the output of the D-A an analog ramp $t, t \geq 0$. When the input $u(kh)$ is a discrete unit step the equivalence remains valid for any analog element placed between the converters.

Solution 8.7:

Backward Euler: In this method s will be replaced with $(z-1)/zh$.

$$K(s) = \frac{1}{s} \Rightarrow K_d(z) = \frac{zh}{z-1}$$

$$K(s) = \frac{s}{s^2+2} \Rightarrow K_d(z) = \frac{\frac{z-1}{zh}}{\left(\frac{z-1}{zh}\right)^2 + 2} = \frac{zh(z-1)}{(1+2h^2)z^2 - 2z + 1}$$

$$K(s) = \frac{s+4}{s+3} \Rightarrow K_d(z) = \frac{\frac{z-1}{zh}+4}{\frac{z-1}{zh}+3} = \frac{(1+4h)z-1}{(1+3h)z-1}$$

$$K(s) = \frac{1}{s(s+8)} \Rightarrow K_d(z) = \frac{1}{\frac{z-1}{zh}\left(\frac{z-1}{zh}+8\right)} = \frac{z^2h^2}{(z-1)((1+8h)z-1)}$$

Tustin: In this method s is replaced with $\frac{2}{h}\frac{z-1}{z+1}$.

$$K(s) = \frac{1}{s} \Rightarrow K_d(z) = \frac{h}{2}\frac{z+1}{z-1}$$

$$K(s) = \frac{s}{s^2+2} \Rightarrow K_d(z) = \frac{\frac{h}{2}(z-1)(z+1)}{\left(1+\frac{h^2}{2}\right)z^2 - 2\left(\frac{h^2}{2}-1\right)z + 1 + \frac{h^2}{2}}$$

$$K(s) = \frac{s+4}{s+3} \Rightarrow K_d(z) = \frac{\left(1+4\frac{h}{2}\right)z + 4\frac{h}{2} - 1}{\left(1+3\frac{h}{2}\right)z + 3\frac{h}{2} - 1}$$

$$K(s) = \frac{1}{s(s+8)} \Rightarrow K_d(z) = \frac{(z+1)^2\left(\frac{h}{2}\right)^2}{(z-1)\left(\left(1+8\frac{h}{2}\right)z + 8\frac{h}{2} - 1\right)}$$

Zero-pole matching: In this method the poles and zeros are mapped by an exponential mapping and then the steady-state gain is matched in one frequency point (generally at $\omega = 0$).

1. There is only one pole at zero which is mapped to 1.

$$K(s) = \frac{1}{s} \Rightarrow K_d(z) = \frac{1}{z-1}\cdot c$$

The constant should be computed to match the gain at one frequency. Here, we can't match gains at DC, since the gain is infinite. Instead, we choose to match the gains at (for example) the crossover frequency, $\omega_c = 1$.

$$|K_d(e^{j\omega_c h})| = |K(j\omega_c)| = \left|\frac{1}{e^{jh}-1}\right|\cdot c = \left|\frac{1}{j}\right| = 1 \qquad \Rightarrow\ c = |e^{jh}-1|$$

Therefore,

$$K_d(z) = \frac{|e^{jh} - 1|}{z - 1}$$

2. There are two poles at $\pm j\sqrt{2}$ and one zero at zero:

$$K_d(z) = c \cdot \frac{z - e^{0h}}{(z - e^{jh\sqrt{2}})(z - e^{-jh\sqrt{2}})} = c \cdot \frac{z - 1}{z^2 - 2z\cos h\sqrt{2} + 1}$$

The transfer functions will already match gains at zeros frequency ($K(0) = K_d(1) = 0$), so we need to choose another frequency to match. In this case, we take $\omega = 1$ (which is as good as any, since the 'best' frequency to take is application dependent):

$$|K(j)| = \left|\frac{j}{-1+2}\right| = |K_d(e^{jh})| = c \cdot \left|\frac{e^{jh} - 1}{e^{j2h} - 2e^{jh}\cos h\sqrt{2} + 1}\right|$$

Therefore,

$$c = \left|\frac{e^{j2h} - 2e^{jh}\cos h\sqrt{2} + 1}{e^{jh} - 1}\right|$$

3. There are one pole at -3 and one zero at -4.

$$K_d(z) = c \cdot \frac{z - e^{-4h}}{z - e^{-3h}}$$

Choose the gain c so that the discrete and continuous models match at DC:

$$|K(0)| = 4/3 = |K_d(1)| = c\frac{1 - e^{-4h}}{1 - e^{-3h}}$$

Therefore,

$$c = \frac{4}{3}\frac{1 - e^{-3h}}{1 - e^{-4h}}$$

4. There are one pole at 0, one pole at -8, and one zero at ∞ (deg of denominator - deg of numerator - 1).

$$K_d(z) = c\frac{z + 1}{(z - 1)(z - e^{-8h})}$$

We can't match the gains at zero frequency, so we choose another one, say $\omega = 1$ for convenience:

$$|K(j)| = \frac{1}{|j + 8|} = 0.124 = |K_d(e^{jh})| = c\left|\frac{e^{jh} + 1}{(e^{jh} - 1)(e^{jh} - e^{-8h})}\right|$$

Therefore,

$$c = 0.124\left|\frac{(e^{jh} - 1)(e^{jh} - e^{-8h})}{e^{jh} + 1}\right|$$

Solution 8.8:

Backward Euler: In this method z is replaced with $1/(1-hs)$. For $h = 1$ we have the following continuous-time model approximation.

$$G_c(s) = \frac{0.04762 + 0.04762(1 - s)}{1 - 0.9048(1 - s)} = \frac{0.09524 - 0.04762s}{0.9048s + 0.0952}$$

Tustin: In this method z is replaced with $\dfrac{1 + hs/2}{1 - hs/2}$. For $h = 1$ we have:

$$G_c(s) = \frac{0.04762(2 + s) + 0.04762(2 - s)}{(2 + s) - 0.9048(2 - s)} = \frac{0.09524}{0.9524s + 0.0952}$$

Solution 8.9: The discretized transfer function of the system to be controlled is computed as follows:

$$H(z) = (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1}{s(s+1)}\right)\right\} = \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1}{s}-\frac{1}{s+1}\right)\right\}$$
$$= \frac{z-1}{z}\mathcal{Z}\left(1-e^{-kh}\right) = \frac{z-1}{z}\left(\frac{z}{z-1}-\frac{z}{z-e^{-h}}\right) = 1-\frac{z-1}{z-e^{-h}} = \frac{1-e^{-h}}{z-e^{-h}}$$

We deduce that the transfer function in closed loop is:

$$\frac{Y(z)}{Y_c(z)} = \frac{K_p\frac{1-e^{-h}}{z-e^{-h}}}{1+K_p\frac{1-e^{-h}}{z-e^{-h}}} = \frac{K_p(1-e^{-h})}{z-e^{-h}+K_p(1-e^{-h})}$$

The BIBO stability of the closed loop is guaranteed if and only if the pole at $z = e^{-h} - K_p(1-e^{-h})$ is in the interior of the unit circle:

$$-1 < e^{-h} - K_p(1-e^{-h}) < 1$$

Where finally we have that if $K_p > 1$ the closed loop system is BIBO stable for:

$$0 < h < -\ln\frac{K_p-1}{K_p+1}$$

If $-1 < K_p \leq 1$ the closed-loop system is BIBO stable for any $h > 0$ and if $K_p \leq -1$ the closed-loop system is unstable for any sampling period.

Solution 8.10: The transfer function of the discretized system to be controlled is computed as follows:

$$H(z) = (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{4}{s(s+2)}\right)\right\} = \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{2}{s}-\frac{2}{s+2}\right)\right\}$$
$$= \frac{z-1}{z}\left(2\frac{z}{z-1}-2\frac{z}{z-e^{-0.05}}\right) = 2\left(1-\frac{z-1}{z-e^{-0.05}}\right)$$
$$= 2\frac{1-e^{-0.05}}{z-e^{-0.05}} \simeq \frac{0.0975}{z-0.95}$$

It follows that the closed loop transfer function is:

$$\frac{Y(z)}{Y_c(z)} = \frac{K_p\frac{0.0975}{z-0.95}}{1+K_p\frac{0.0975}{z-0.95}} = \frac{0.0975K_p}{z-0.95+0.0975K_p}$$

For stability we require that:

$$-1 < -0.95 + 0.0975K_p < 1 \quad \Leftrightarrow \quad -0.513 < K_p < 20$$

Taking only positive gains, we have: $0 < K_p < 20$. Therefore the ultimate gain is $K_u = 20$.

Solution 8.11: Using the backward transformation we obtain:

$$K(q^{-1}) = K_P + \frac{K_I h}{1-q^{-1}} = \frac{(K_P + K_I h) - K_P q^{-1}}{1-q^{-1}}$$

Note that $u(t) = K(q^{-1})[r(t) - y(t)]$ or:

$$(1-q^{-1})u(t) = [(K_P + K_I h) - K_P q^{-1}]r(t) - [(K_P + K_I h) - K_P q^{-1}]y(t)$$

Comparing the above equation with the control law in an RST controller

$$S(q^{-1})u(t) = T(q^{-1})r(t) - R(q^{-1})y(t)$$

leads to:

$$R(q^{-1}) = (K_P + K_I h) - K_P q^{-1}$$
$$S(q^{-1}) = 1 - q^{-1}$$
$$T(q^{-1}) = (K_P + K_I h) - K_P q^{-1}$$

Solution 8.12: In the first step $G(z)$ should be converted to $G(q^{-1})$ by replacing z with q:

$$G(q^{-1}) = \frac{0.5q - 0.4}{q^2 - 1.2q + 0.35} = \frac{0.5q^{-1} - 0.4q^{-2}}{1 - 1.2q^{-1} + 0.35q^{-2}}$$

Therefore, $n_A = 2$, $n_B = 2$.

1. We have $n_R = n_A - 1 = 1$ and $n_S = n_B - 1 = 1$, therefore, the Bezout equation is:

$$(1 - 1.2q^{-1} + 0.35q^{-2})(1 + s_1 q^{-1}) + (0.5q^{-1} - 0.4q^{-2})(r_0 + r_1 q^{-1}) = 1 - 0.7q^{-1}$$

By making equal the coefficients of the same power of q on both sides, we obtain:

$$\begin{aligned} -1.2 + s_1 + 0.5r_0 &= -0.7 \\ 0.35 - 1.2s_1 - 0.4r_0 + 0.5r_1 &= 0 \qquad \Rightarrow s_1 = 1.333, r_0 = -1.666, r_1 = 1.1667 \\ 0.35s_1 - 0.4r_1 &= 0 \end{aligned}$$

Then $T(q^{-1}) = P(1)/B(1) = 0.3/0.1 = 3$.

2. To have an integrator in the controller we should take $H_S(q^{-1}) = 1 - q^{-1}$ and define

$$A'(q^{-1}) = A(q^{-1})H_S(q^{-1}) = 1 - 2.2q^{-1} + 1.55q^{-2} - 0.35q^{-3}$$

Then $n_{R'} = 2$ and we should solve:

$$(1 - 2.2q^{-1} + 1.55q^{-2} - 0.35q^{-3})(1 + s_1 q^{-1}) + (0.5q^{-1} - 0.4q^{-2})(r_0 + r_1 q^{-1} + r_2 q^{-2}) = 1 - 0.7q^{-1}$$

Which leads to the following matrix equality using the Sylvester matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2.2 & 1 & 0.5 & 0 & 0 \\ 1.55 & -2.2 & -0.4 & 0.5 & 0 \\ -0.35 & 1.55 & 0 & -0.4 & 0.5 \\ 0 & -0.35 & 0 & 0 & -0.4 \end{bmatrix} \begin{bmatrix} 1 \\ s'_0 \\ r_0 \\ r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.7 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution to this equation is:

$$\begin{aligned} R(q^{-1}) &= 21.667 - 26.833q^{-1} + 8.1667q^{-2} \\ S(q^{-1}) &= 1 - 10.333q^{-1} + 9.333q^{-2} \end{aligned}$$

For an RST controller with integrator the polynomial $T(q^{-1})$ will be $T(q^{-1}) = R(1) = 3$.

Solution 8.13: For the MRC problem, we should check first if the zeros of $B^\star(q^{-1})$ are inside the unit circle. We have: $B^\star(q^{-1}) = 0.5 - 0.4q^{-1}$ and $d = 1$, therefore, $0.5z - 0.4 = 0$ gives $z = 0.8$ which is inside the unit circle. Then, we should compute the closed-loop polynomial $P_d(q^{-1})$. We have:

$$\begin{aligned} p_1 &= -2e^{-\zeta\omega_n h} \cos\left(\omega_n h\sqrt{1-\zeta^2}\right) = -0.7417 \\ p_2 &= e^{-2\zeta\omega_n h} = 0.2019 \end{aligned}$$

Therefore, $P_d(q^{-1}) = 1 - 0.7417q^{-1} + 0.2019q^{-2}$ and we choose $P(q^{-1}) = P_d(q^{-1})B^\star(q^{-1})$.

1. For RST controller design we should solve the following equation:

$$A(q^{-1})S(q^{-1}) + q^{-1}B^\star(q^{-1})R(q^{-1}) = P_d(q^{-1})B^\star(q^{-1})$$

This equation has a solution if $S(q^{-1}) = S'(q^{-1})B^\star(q^{-1})$, that leads to:

$$A(q^{-1})S'(q^{-1}) + q^{-1}R(q^{-1}) = P_d(q^{-1})$$

We have $n_R = n_A - 1 = 1$ and $n_{S'} = 1 - 1 = 0$. Thus $R(q^{-1}) = r_0 + r_1 q^{-1}$ and $S'(q^{-1}) = 1$:

$$1 - 1.2q^{-1} + 0.35q^{-2} + r_0 q^{-1} + r_1 q^{-2} = 1 - 0.7417q^{-1} + 0.2019q^{-2}$$

Solving the equation gives $r_0 = 0.4583$ and $r_1 = -0.1481$. So the final RST controller is:

$$\begin{aligned} R(q^{-1}) &= 0.4583 - 0.1481q^{-1} \\ S(q^{-1}) &= B^\star(q^{-1})S'(q^{-1}) = 0.5 - 0.4q^{-1} \\ T(q^{-1}) &= 1 - 0.7417q^{-1} + 0.2019q^{-2} \end{aligned}$$

2. To add a fixed term $H_R(q^{-1}) = 1 + q^{-1}$ in the controller, we define $R(q^{-1}) = H_R(q^{-1})R'(q^{-1})$ and we should solve the following Bezout equation:

$$A(q^{-1})S'(q^{-1}) + q^{-1}(1 + q^{-1})R'(q^{-1}) = P_d(q^{-1})$$

In order to have a solution $n_{S'} = 1$ and so $S'(q^{-1}) = 1 + s'_1 q^{-1}$ and $n_{R'} = n_A - 1 = 1$, therefore:

$$(1 - 1.2q^{-1} + 0.35q^{-2})(1 + s'_1 q^{-1}) + q^{-1}(1 + q^{-1})(r'_0 + r'_1 q^{-1}) = 1 - 0.7417q^{-1} + 0.2019q^{-2}$$

which leads to the following system of linear equations:

$$\begin{aligned} -1.2 + s'_1 + r'_0 &= -0.7417 \\ -1.2s'_1 + 0.35 + r'_0 + r'_1 &= 0.2019 \quad \Rightarrow \quad s'_1 = 0.2378, r'_0 = 0.2205, r'_1 = -0.0832 \\ 0.35s'_1 + r'_1 &= 0 \end{aligned}$$

So the final RST controller is:

$$\begin{aligned} R(q^{-1}) &= H_R(q^{-1})R'(q^{-1}) = (1 + q^{-1})(0.2205 - 0.0832q^{-1}) = 0.2205 + 0.1373q^{-1} - 0.0832q^{-2} \\ S(q^{-1}) &= B^\star(q^{-1})S'(q^{-1}) = (0.5 - 0.4q^{-1})(1 + 0.2378q^{-1}) = 0.5 - 0.2811q^{-1} - 0.0951q^{-2} \\ T(q^{-1}) &= 1 - 0.7417q^{-1} + 0.2019q^{-2} \end{aligned}$$

Solution 8.14: The Bezout equation is:

$$A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1}) = A(q^{-1})$$

We take $R(q^{-1}) = R'(q^{-1})A(q^{-1})$ and simplify the Bezout equation as follows:

$$S(q^{-1}) + B(q^{-1})R'(q^{-1}) = 1$$

This equation has many solutions, choosing $R'(q^{-1}) = r_0$ (to obtain the lowest order solutions), we obtain:

$$S(q^{-1}) = 1 - B(q^{-1})r_0 = 1 - (0.5q^{-1} - 0.4q^{-2})r_0$$

For any value of r_0 we have a valid solution. In order to have an integrator in the controller, we pose $S(1) = 0$ that leads to:

$$1 - 0.5r_0 + 0.4r_0 = 0 \quad \Rightarrow \quad r_0 = 10$$

Therefore:

$$\begin{aligned} R(q^{-1}) &= R'(q^{-1})A(q^{-1}) = 10 - 12q^{-1} + 3.5q^{-2} \\ S(q^{-1}) &= 1 - 5q^{-1} + 4q^{-2} \\ T(q^{-1}) &= R(1) = 1.5 \end{aligned}$$

Solution 8.15:

1. $H(z)$ using the ZOH method:

$$H(z) = z^{-2}\frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{2.6}{s(s+1.3)}\right\}\right\} = z^{-2}\frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{-2}{s+1.3}\right\}\right\}$$
$$= z^{-2}\frac{z-1}{z}\left[\frac{2z}{z-1}-\frac{2z}{z-0.85}\right] = \frac{0.3z^{-2}}{z-0.85} = \frac{0.3z^{-3}}{1-0.85z^{-1}}$$

2. From $H(z)$ we obtain $A(q^{-1}) = 1-0.85q^{-1}$, $B(q^{-1}) = 0.3q^{-3}$. In order to have an integrator we take $H_S(q^{-1}) = 1-q^{-1}$ and

$$A'(q^{-1}) = (1-0.85q^{-1})(1-q^{-1}) = 1-1.85q^{-1}+0.85q^{-2}$$

Then we should solve the following Bezout equation:

$$A'(q^{-1})S'(q^{-1}) + B(q^{-1})R(q^{-1}) = 1-0.8q^{-1}$$

We have $n_R = n_{A'} - 1 = 1$ and $n_{S'} = n_B - 1 = 2$, which leads to $R(q^{-1}) = r_0 + r_1q^{-1}$ and $S'(q^{-1}) = 1 + s_1q^{-1} + s_2q^{-2}$.

$$(1-1.85q^{-1}+0.85q^{-2})(1+s_1q^{-1}+s_2q^{-2}) + 0.3q^{-3}(r_0+r_1q^{-1}) = 1-0.8q^{-1}$$

$$s_1 - 1.85 = -0.8 \quad\Rightarrow\quad s_1 = 1.05$$
$$0.85 - 1.85s_1 + s_2 = 0 \quad\Rightarrow\quad s_2 = 1.05\times1.85-0.85 = 1.09$$
$$-1.85s_2 + 0.85s_1 + 0.3r_0 = 0 \quad\Rightarrow\quad r_0 = (1.85\times1.09-0.85\times1.05)3.33 = 3.76$$
$$0.85s_2 + 0.3r_1 = 0 \quad\Rightarrow\quad r_1 = (-0.85\times1.09)3.33 = -3.095$$

Therefore:

$$R(q^{-1}) = 3.76 - 3.095q^{-1}$$
$$S(q^{-1}) = (1+1.05q^{-1}+1.09q^{-2})(1-q^{-1})$$
$$T(q^{-1}) = R(1) = 0.6667$$

Solution 8.16: The desired bandwidth is 2 rad/s then:

$$\tau = 0.5 \quad\Rightarrow\quad M(s) = \frac{1}{0.5s+1} = \frac{2}{s+2} \quad\Rightarrow\quad H_m(z) = (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{2}{s(s+2)}\right)\right\}$$

$$H_m(z) = (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1}{s}+\frac{-1}{(s+2)}\right)\right\} = (1-z^{-1})\left(\frac{z}{z-1}-\frac{z}{z-e^{-2h}}\right) = \frac{0.18}{z-0.82}$$

Then we compute the desired closed-loop poles for regulation based on $\omega_n = 4, \zeta = 0.8$:

$$p_1 = -2e^{-\zeta\omega_n h}\cos(\omega_n h\sqrt{1-\zeta^2}) = -1.41 \quad ; \quad p_2 = e^{-2\zeta\omega_n h} = 0.527$$

Then we compute $H(q^{-1})$:

$$H(z) = \frac{4(0.8-z)}{z(0.5-z)(1.8-z)} = \frac{3.2-4z}{z^3-2.3z^2+0.9z} \quad\Rightarrow\quad H(q^{-1}) = \frac{-4q^{-2}+3.2q^{-3}}{1-2.3q^{-1}+0.9q^{-2}}$$

$$\Rightarrow\quad A(q^{-1}) = 1-2.3q^{-1}+0.9q^{-2} \quad , \quad B^\star(q^{-1}) = -4+3.2q^{-1} \quad , \quad d = 2$$

Note that $B^\star(q^{-1})$ has a root inside the unit circle at 0.8. We take $H_S(q^{-1}) = 1-q^{-1}$. Therefore, $n_{S'} = 1$ and $n_R = n_A + n_{H_S} - 1 = 2$.

Then we write the Bezout equation:

$$A(q^{-1})(1-q^{-1})S'(q^{-1})B^{\star}(q^{-1}) + q^{-2}B^{\star}(q^{-1})R(q^{-1}) = P(q^{-1})B^{\star}(q^{-1})$$

$$(1-3.3q^{-1}+3.2q^{-2}-0.9q^{-3})(1+s_1'q^{-1}) + q^{-2}(r_0+r_1q^{-1}+r_2q^{-2}) = 1-1.41q^{-1}+0.527q^{-2}$$

$$\begin{aligned}
-3.3+s_1' = -1.41 \quad &\Rightarrow \quad s_1' = 1.89\\
3.2-3.3s_1'+r_0 = 0.527 \quad &\Rightarrow \quad r_0 = 3.564\\
-0.9+3.2s_1'+r_1 = 0 \quad &\Rightarrow \quad r_1 = -5.148\\
-0.9s_1'+r_2 = 0 \quad &\Rightarrow \quad r_2 = 1.7
\end{aligned}$$

And the final RST controller is given as:

$$\begin{aligned}
&R(q^{-1}) = 3.564-5.148q^{-1}+1.7q^{-2}\\
&S(q^{-1}) = (1+1.89q^{-1})(1-q^{-1})(3.2q^{-1}-4) = -4-6.76q^{-1}+4.712q^{-2}+6.05q^{-3}\\
&T(q^{-1}) = 1-1.41q^{-1}+0.527q^{-2}
\end{aligned}$$