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AG II - Schemes and sheaves

Solutions – week 13

Exercise 1. *Separable extensions and differentials.* Let k be a field and l a finite extension. Show that $\Omega^1_{l|k} = 0$ if and only if l is a separable extension.

Solution key. If l is separable and finite then $l = k(\alpha)$ for some algebraic and separable element α. But then $l = k[t]/(f(t))$ and $f'(t)$ is not zero in the quotient by separability. This concludes one way by the conormal sequence. For the other direction, if l is not separable, then there is some $\alpha \in l$ such that $l'(\alpha) = l$ for some sub-extension l' such that the extension is not separable implying that $l = l'[t]/(f(t))$ with the derivative vanishing in this quotient. Therefore $\Omega_{l|l'} = l$ by the conormal sequence. But if $\Omega_{l|k} = 0$ then $\Omega_{l|l'} = 0$ by the fundamental sequence of cotangent sheaves, which is a contradiction.

□

Exercise 2. *Derivations on an elliptic curve.* Let R be a ring, $P = R[x_1, \dots, x_n]$ and $P \to A$ a surjection, with kernel I. Recall that by the conormal sequence, if $d\colon I/I^2 \to \bigoplus_{i=1}^n Adx_i$ is given by sending a polynomial to the image of it's derivative then we have an exact sequence

$$I/I^2 \to \bigoplus_{i=1}^{n} Adx_i \to \Omega^1_{A|R} \to 0.$$

We denote by $\mathrm{T}^1_{A|R} = \mathrm{Hom}_A(\Omega^1_{A|R}, A) = \mathrm{Der}_R(A, A)$, the *$A$-module of R-derivations of A.*

(1) Let
$$E = \mathrm{Proj}(\mathbb{C}[X, Y, Z]/(Y^2Z - (X^3 + Z^3))).$$
Denote by x, y the images of $\frac{X}{Z}, \frac{Y}{Z}$ in $A_Z := \mathcal{O}_E(D_+(Z))$ and s, t the images of $\frac{X}{Y}, \frac{Z}{Y}$ in $A_Y := \mathcal{O}_E(D_+(Y))$. Show using the sequence recalled above that (meaning that any derivation is a scalar multiplication of the written generator)

$$\mathrm{T}^1_{A_Z|\mathbb{C}} = A_Z(2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y}) \quad \mathrm{T}^1_{A_Y|\mathbb{C}} = A_Y((3t^2-1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}).$$

(2) Moreover show that the generators displayed above agree on the intersection $D_+(YZ)$, giving a non-vanishing global section π of $\mathrm{T}_{E|\mathbb{C}} := \mathrm{Hom}_{\mathcal{O}_E}(\Omega^1_{E|\mathbb{C}}, \mathcal{O}_E)$ implying that

$$\mathrm{T}_{E|\mathbb{C}} = \mathcal{O}_{E|\mathbb{C}}\pi.$$

Solution key. (1) We write functions on $D_+(Z)$ and $D_+(Y)$. They are

$$\mathbb{C}[x', y']/(y'^2 - (x'^3 + 1)) \quad \mathbb{C}[s', t']/(t' - t'^3 - s'^3)$$

where x', y', s', t' denotes $\frac{X}{Z}, \frac{Y}{Z}$ and $\frac{X}{Y}, \frac{Z}{Y}$ before taking the quotient. By the conormal sequence we have

$$\Omega_{A_Z|k} = \frac{A_Z dx' \oplus A_Z dy'}{-3x^2 dx' + 2y dy'} \quad \Omega_{A_Y|k} = \frac{A_Y ds' \oplus A_Y dt'}{-3s^2 ds' + (1-3t^2)dt'}$$

We are interested in the dual of both these modules. We see the dual as a submodule of the dual of $A_Z dx' \oplus A_Z dy'$ and $A_Y ds' \oplus A_Y dt'$ respectively. Using the identification with derivations, we write the dual basis (dx', dy') and (ds', dt') as $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and $(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})$. The claimed generators are indeed in this submodule. We want to show that they generate. So let $f_1, f_2 \in A_Y$ such that

$$f_1(-3x^2) + f_2 2y = 0.$$

So $f_1 3x^2 = f_2 2y$. But $A_Z/y = \mathbb{C}[x']/(x'^3 + 1)$ and $3x^2$ is invertible in this ring (it's not a root of the polynonial). So we see that $2y \mid f_1$. Also $A_Y/x^2 = \mathbb{C}[x', y']/(x^2, y^2 - 1)$ and a similar argument holds to conclude that $3x^2 \mid f_2$. So we have $f_1 = 2y\lambda_1$ and $f_2 = 3x^2\lambda_2$. Therefore $2y\lambda_1 3x^2 = 3x^2\lambda_2 2y$. We can simplify to get $\lambda_1 = \lambda_2$, which concludes. The reasoning for the second module is similar.

Also, note that because rings that we are dealing with are integral, we necessarily have that the map $A_Z \to A_Z\frac{\partial}{\partial x} \oplus A_Z\frac{\partial}{\partial y}$ sending 1 to the generator is injective. Same holds for the second module. We have therefore concluded that

$$\mathrm{T}^1_{A_Z|\mathbb{C}} = A_Z(2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y}) \quad \mathrm{T}^1_{A_Y|\mathbb{C}} = A_Y((3t^2-1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t})$$

are free sheaves of rank 1.

(2) It suffices to show that both derivations agree on the intersection. Note that $x = st^{-1}$ and $y = t^{-1}$. Now,

$$\left((3t^2-1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}\right)(st^{-1}) = \frac{3t^2-1}{t} + \frac{3s^3}{t^2}.$$

But this equals, because $s^3 = t - t^3$, to $2/t = 2y$. Also

$$\left((3t^2-1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}\right)(t^{-1}) = \frac{3s^2}{t^2} = 3x^2,$$

which concludes that derivations indeed agree on the intersection.

Last statement follows because $T^1_{E|k}$ is an invertible sheaf by the first point and that we found a global nowhere vanishing global section.

□

Exercise 3. *Relative* Spec. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. This means that it is a sheaf $\mathcal{O}_S$-algebras which is quasi-coherent as an $\mathcal{O}_S$-module.

(1) Let $V \subset U \subset S$ two open affines. Show that the diagram

$$\begin{array}{ccc} \operatorname{Spec}(\mathcal{A}(V)) & \longrightarrow & \operatorname{Spec}(\mathcal{A}(U)) \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \end{array}$$

is cartesian.

(2) Let $X = \bigcup U_i$ be an affine cover. Deduce that we can glue the schemes $(\operatorname{Spec}(\mathcal{A}(U_i)))$ to an S-scheme

$$\underline{\operatorname{Spec}}_S(\mathcal{A}) \to S.$$

(3) Show that $\underline{\operatorname{Spec}}_S(\mathcal{A})$ satisfies the following universal property in the category of S-schemes. If $f\colon T \to S$ is an S-scheme then a S-morphism $T \to \underline{\operatorname{Spec}}_S(\mathcal{A})$ is the same as a morphism of $\mathcal{O}_T$-algebras $f^*\mathcal{A} \to \mathcal{O}_T$. Deduce that $\underline{\operatorname{Spec}}_S(\mathcal{A})$ is independent of the affine cover for the construction.

(4) Let $f\colon X \to Y$ be an affine morphism of schemes. Show that there is a natural isomorphism of Y-schemes $X \cong \underline{\operatorname{Spec}}_Y(f_*\mathcal{O}_X)$.

(5) Let $\mathcal{E}$ be a locally free sheaf of finite rank on S. We define

$$\mathbb{V}(\mathcal{E}) = \underline{\operatorname{Spec}}_S(\operatorname{Sym}(\mathcal{E}^\vee))$$

where the $\mathcal{O}_S$-algebra $\operatorname{Sym}(\mathcal{E}^\vee)$ denotes the free $\mathcal{O}$-algebra on $\mathcal{E}^\vee$.[1] Show that a S-morphism from $f\colon T \to S$ to $\mathbb{V}(\mathcal{E})$ is the same as a global section of $f^*(\mathcal{E})$, *i.e* an element of $f^*(\mathcal{E})(T)$.

(6) Show that there is always a canonical section of $p\colon \mathbb{V}(\mathcal{E}) \to S$ which correspond to $0 \in \mathcal{E}(S)$ which defines a closed subscheme of $\mathbb{V}(\mathcal{E})$ isomorphic to S. We call this closed subscheme the *zero section of* $\mathbb{V}(\mathcal{E})$.

Solution key. (1) We may cover V by principal opens affine to prove the isomorphism locally. In this case it is clear that it follows from quasi-coherence.

(2) Using the above we see that $\operatorname{Spec}(\mathcal{A}(U_{ij})) \to \operatorname{Spec}(\mathcal{A}(U_i))$ are open immersions, being locally pullbacks of open immersions. It follows that we can glue this to a scheme.

(3) Cover S by affine schemes U_i. It induces an open cover of T by open subschemes T_i. Suppose we are given $f\colon T \to \underline{\operatorname{Spec}}_S(\mathcal{A})$. Note that this corresponds by gluing to a collection of maps of U_i-schemes $f\colon T_i \to \operatorname{Spec}(\mathcal{A}(U_i))$ that appropriately glues. This correspond one to one to a collection of $\mathcal{O}(U_i)$-algebra maps $\mathcal{A}(U_i) \to \mathcal{O}(T_i)$ which correspond one to one to morphisms of $\mathcal{O}_S$-algebras $\mathcal{A} \to f_*\mathcal{O}_T$, from which the claim follows by adjunction.

(4) As f is affine, note that $f_*\mathcal{O}_X$ is quasi-coherent. Note also that for an open affine $U \subset Y$, we have a natural identification $f^{-1}(U) = \operatorname{Spec}(\mathcal{O}(f^{-1}(U))) = \operatorname{Spec}(f_*\mathcal{O}_X(U))$ from which the claim follows.

(5) By the above, such a morphism is the same as the data of an $\mathcal{O}_T$-algebra morphism $\operatorname{Sym}(f^*\mathcal{E}^\vee) \to \mathcal{O}_T$. Therefore this the same as an

[1] It's a gluing of the usual construction in liner algebra.

$\mathcal{O}_T$-module morphism $f^*\mathcal{E}^\vee \to \mathcal{O}_T$. By duality, this is the same as a section of $f^*\mathcal{E}$.

(6) Affine locally on S, say on some affine $\operatorname{Spec}(R)$, we can assume that $\mathcal{E} \cong \widetilde{R^n}$ is finite free, and the zero section correspond to the origin of $\mathbb{A}^n_R$.

□

Remark. If $S = \operatorname{Spec}(k)$ and V a finite-dimensional vector space, then $\mathbb{V}(V)$ is the scheme-theoretic incarnation of the k-vector space V.

Exercise 4. *Projective bundles.* Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. Let $\mathcal{E}$ be a locally free sheaf of finite rank on S.

(1) Let $V \subset U \subset S$ two open affines. Show that the diagram

$$\begin{array}{ccc} \operatorname{Proj}(\mathcal{A})(V)) & \longrightarrow & \operatorname{Proj}(\mathcal{A})(U)) \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \end{array}$$

is cartesian.

(2) Let $X = \bigcup U_i$ be an affine cover. Deduce that we can glue the schemes $(\operatorname{Proj}(\mathcal{A})(U_i)))$ to an S-scheme (*the relative* Proj)

$$\pi \colon \underline{\operatorname{Proj}}(\mathcal{A}) \to S.$$

When $\mathcal{A} = \operatorname{Sym}(\mathcal{E}^\vee)$ we denote $\underline{\operatorname{Proj}}(\operatorname{Sym}(\mathcal{E}^\vee)) = \mathbb{P}(\mathcal{E})$, the *projective bundle associated to* $\mathcal{E}$.

(3) Show that $\mathbb{P}(\mathcal{E})$ satisfies the following universal property in the category of S-schemes. If $f \colon T \to S$ is an S-scheme then a S-morphism $T \to \mathbb{P}(\mathcal{E})$ is the same as a sub-line bundle[2] $\mathcal{L} \subset f^*\mathcal{E}$.
Hint: Show that the line bundles $\mathcal{O}(1)$ on $\operatorname{Proj}(\operatorname{Sym}(\mathcal{E}^\vee)(U))$ glue naturally to a line bundle $\mathcal{O}(1)$ with a surjection

$$\pi^*\mathcal{E}^\vee \to \mathcal{O}(1).$$

The identity correspond therefore to the dual inclusion $\mathcal{O}(-1) \subset \pi^*\mathcal{E}$. Recall that for locally free sheaves of finite rank, pullback and dual naturally commute.

(4) Show that the surjection

$$\operatorname{Sym}(\mathcal{E}^\vee \oplus \mathcal{O}_S) \to \operatorname{Sym}(\mathcal{E}^\vee)$$

induces a closed immersion

$$\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E} \oplus \mathcal{O})$$

and that the open complement identifies to $\mathbb{V}(\mathcal{E})$, leading to an open-closed decomposition

$$\mathbb{P}(\mathcal{E} \oplus \mathcal{O}) = \mathbb{V}(\mathcal{E}) \sqcup \mathbb{P}(\mathcal{E}).$$

Remark. This generalizes the open closed decomposition $\mathbb{P}^{n+1}_k = \mathbb{A}^{n+1}_k \sqcup \mathbb{P}^n_k$. We can therefore interpret $\mathbb{P}(\mathcal{E} \oplus \mathcal{O})$ as a compactification

[2] a subsheaf which is a line bundle, and such that $f^*\mathcal{E}/\mathcal{L}$ is locally free.

of $\mathbb{V}(\mathcal{E})$ where we add an ∞-point to each line in $\mathbb{V}(\mathcal{E})$, namely the corresponding point in $\mathbb{P}(\mathcal{E})$.

(5) Show that $\mathcal{O} \subset \mathcal{E} \oplus \mathcal{O}$ defines a section of $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \to S$ which leads to an open-closed decomposition

$$\mathbb{V}(\mathcal{O}(1)) \sqcup S = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}).$$

Remark. This generalizes

$$\mathbb{P}^{n+1}_k = \left(\bigcup_{i=0}^{n} D_+(x_i) \right) \sqcup [0 : \ldots 0 : 1].$$

Solution key. (1) Analogous to the above.

(2) Same.

(3) We give a proof that does not use the proposition about morphism to $\mathbb{P}^n_A$ case. In fact, it is a reformulation of this proof in other terms and a more general setup. We try take profit as much as we can of the known properties of the Proj construction, in particular it's functoriality studied in a previous exercise.

(a) *Some facts on* $\underline{\mathrm{Proj}}_S$. Let $\mathcal{F}$ be a finite locally free sheaf on S. We denote by $\pi\colon \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{F})) \to S$ the structure map.

Claim. We have a canonical map $\mathcal{F} \to \pi_*\mathcal{O}(1)$ which is an isomorphism. By adjunction, we get a canonical map $\pi^*\mathcal{F} \to \mathcal{O}(1)$. This last map is surjective.

Proof. Note first that for $\mathcal{F} = \mathcal{O}_S^{\oplus n}$, the claim follows from the calculation of the global sections of $\mathcal{O}(1)$ on the projective space, namely it is it's natural reinterpretation. We are going construct a map $\mathcal{F} \to \pi_*\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{F}))}(1)$ which is functorial in $\mathcal{F}$ and this will allow to conclude as in exercise 2, week 9, the exercise on duals.

We may define the natural map $\mathcal{F} \to \pi_*\mathcal{O}(1)$ affine locally on S, because it will readily glue. So say S is affine, where $\mathcal{F}$ is finite free. We may cover $\mathrm{Proj}(\mathrm{Sym}(\mathcal{F}))$ by $D_+(f)$ for $f \in \mathcal{F}(S)$. We have

$$\mathcal{O}(1)(D_+(f)) = \mathrm{Sym}(\mathcal{F})(1)_{(f)}.$$

Therefore we can define the natural map $\mathcal{F} \to \pi_*\mathcal{O}(1)$ by sending $g \in \mathcal{F}(S)$ to the unique global section of $\mathcal{O}(1)$ that restricts locally to $(g \in \mathcal{O}(1)(D_+(f)))_{f\in\mathcal{F}}$. Because this map is natural[3] we can conclude using the local free case that we already know as explained above.

Affine locally on S, and on an open $D_+(f)$ the surjection $\pi^*\mathcal{F} \to \mathcal{O}(1)$ reads as

$$\mathrm{Sym}(\mathcal{F})_{(f)} \otimes \mathcal{F} \to \mathrm{Sym}(\mathcal{F})(1)_{(f)} \quad 1 \otimes f \to f$$

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[3] If some reader want some language, we are defining a natural transformation between the identity functor on locally free sheaves and the functor sending $\mathcal{F}$ to $\pi_*\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{F}))}(1)$.

which is surjective.
Taking duals of the previous point, we get,
Corollary. *On $\mathbb{P}(\mathcal{E})$, we have a natural inclusion $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \subset \pi^*\mathcal{E}$.*
See the next exercise for more on this inclusion.

□

(b) Let $\mathcal{L}$ be a line bundle on S.
Claim. *The map $\pi\colon \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{L})) \to S$ is an equality. Moreover if $s \in \mathcal{L}(S)$ is a global section, then $D_+(s)$ in $\underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{L}))$ correspond to $D(s)$ in S. Moreover $\mathcal{L}$ corresponds to $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{L}))}(1)$ by point (a).*
We may work locally on S where $\mathcal{L} = \mathcal{O}_S t$ for a generator $t \in \mathcal{L}(S)$. So $\mathrm{Sym}(\mathcal{L}) = \mathcal{O}_S[t]$. But then $\underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{L})) = \underline{\mathrm{Spec}}_S(\mathcal{O}_S[t]_{(t)}) = \underline{\mathrm{Spec}}_S(\mathcal{O}_S)$. Indeed, the natural inclusion $\mathcal{O}_S \subset \mathcal{O}[t]_{(t)}$ is an equality. The claim about $D_+(s)$ and $D(s)$ also follows from the previous inspection.

(c) We now proceed to the proof of the statement of the exercise. Let $T \xrightarrow{f} S$ be an S-scheme. We define a functor $\mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Set}$

$$P(T \xrightarrow{f} S) = \{\mathcal{L} \subset f^*\mathcal{E} \ \mid \mathcal{L} \text{ is a line bundle, and } f^*\mathcal{E}/\mathcal{L} \text{ is locally free}\}$$

The functoriality is defined as follows. Let $f\colon T \to S$ and $f'\colon T' \to S$ be S-schemes. If $g\colon T' \to T$ is a morphism of S-schemes we define $P(g)$ to be $\mathrm{Im}(g^*\mathcal{L} \to f'^*\mathcal{E})$ that we may abbreviate as $g^*\mathcal{L} \subset f'^*\mathcal{E}$.
We may write $P(T)$ and let $f\colon T \to S$ implicit. We want to show that there is a natural bijection

$$P(T) \cong \mathrm{Sch}_S(T, \mathbb{P}(\mathcal{E})).$$

As $T \times_S \mathbb{P}(\mathcal{E}) = \mathbb{P}(f^*\mathcal{E})$, we have $\mathrm{Sch}_S(T, \mathbb{P}(\mathcal{E})) \cong \mathrm{Sch}_T(T, \mathbb{P}(f^*\mathcal{E}))$ by sending a map $T \to \mathbb{P}(f^*\mathcal{E})$ to $T \to \mathbb{P}(f^*\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ so we can suppose that $T = S$ and $f = \mathrm{id}$. We now define a map

$$\alpha\colon P(S) \to \mathrm{Sch}_S(S, \mathbb{P}(\mathcal{E}))$$

by sending $\mathcal{L} \mapsto (S = \mathbb{P}(\mathcal{L}) \to \mathbb{P}(\mathcal{E}))$. The first arrow is the equality explained in point (b) and the second arrow comes from the surjection $\mathcal{E}^\vee \to \mathcal{L}^\vee$ (dual to the given inclusion $\mathcal{L} \subset \mathcal{E}$) and the functoriality of $\underline{\mathrm{Proj}}_S$. We want to show that this is a bijection. To this end we define an inverse map β. Given a section $g\colon S \to \mathbb{P}(\mathcal{E})$, we get using the natural inclusion $\mathcal{O}(-1) \subset \pi^*\mathcal{E}$ an inclusion $g^*\mathcal{O}(-1) \subset \mathcal{E}$. This is our β.

- Note that $\beta \circ \alpha = \mathrm{id}$ because the natural surjection $\pi^*\mathcal{E}^\vee \to \mathcal{O}(1)$ pullback *via* $(S = \mathbb{P}(\mathcal{L}) \to \mathbb{P}(\mathcal{E}))$ to $\mathcal{E}^\vee \to \mathcal{L}^\vee$ by functoriality of Proj and (a) and (b) above.
- We now show that $\alpha \circ \beta = \mathrm{id}$, this will conclude the proof. Let $g\colon S \to \mathbb{P}(\mathcal{E})$ a section, meaning an S-scheme map, meaning $\pi \circ g = \mathrm{id}$. We want to show that the following

diagram commutes

$$\begin{array}{ccc} & & \mathbb{P}(g^*\mathcal{O}(-1)) \\ & \mathrel{\rotatebox{30}{$=$}} & \downarrow \\ S & \xrightarrow{g} & \mathbb{P}(\mathcal{E}) \end{array}$$

which basically means that we can identify g to the map obtained by functoriality of Proj. This is actually a direct consequence of the definition of pullback os $\mathcal{O}_S$-modules and the functoriality of Proj, but we try to write it down carefully in what follows.
To see that it is the case, we may work affine locally on S; let's write then $\mathcal{F}(S) = M$. The map induced by Proj comes from the pullback by g of the natural map $\pi^* M^\vee \to \mathcal{O}(1)$, which is a map $M^\vee \to g^*\mathcal{O}(1)$. We denote the image by this map of some $\phi \in M^\vee$ by $g^*\phi$. The map g is given locally by compatible ring maps

$$g^\sharp \colon \operatorname{Sym}(M^\vee)_{(\phi)} \to \mathcal{O}_S((D(g^*\phi))).$$

Note that on $D(g^*\phi)$ the line bundle $g^*\mathcal{O}(1)$ is trivial. Indeed on $D_+(\phi)$ it is equal to $\operatorname{Sym}(M^\vee)_{(\phi)}\phi$, so the pullback is equal to $\mathcal{O}_S(D(g^*\phi))g^*\phi$. Also note that $g^\sharp(\frac{\psi}{\phi})g^*\phi = g^*\psi$ in $g^*\mathcal{O}(1)(D(g^*\phi))$ because $\frac{\psi}{\phi}\phi = \psi$ in $\operatorname{Sym}(M^\vee)_{(\phi)}\phi = \mathcal{O}(1)(D_+(\phi))$. But now, it immediately follows that the map induced by Proj is locally given by

$$\operatorname{Sym}(M^\vee)_{(\phi)} \to \operatorname{Sym}(g^*\mathcal{O}(1))_{(g^*\phi)} = \mathcal{O}_S(D(g^*\phi))[g^*\phi]_{(g^*\phi)} = \mathcal{O}_S(D(g^*\phi))$$

which sends $\frac{\psi}{\phi}$ to $\frac{g^*\psi}{g^*\phi} = \frac{g^\sharp(\frac{\psi}{\phi})g^*\phi}{g^*\phi} = g^\sharp(\frac{\psi}{\phi})$, which concludes.
Remark. If $S = \operatorname{Spec}(k)$ and V a finite-dimensional vector space, then $\mathbb{P}(V)$ is the scheme-theoretic incarnation of the projective space of V.
Remark. If $\mathcal{E} = \mathcal{O}_S^{\oplus n+1}$, then we get a generalization of statement seen in class over an affine base of the universal property of $\mathbb{P}^n_S$. Let $T \to \mathbb{P}^n_S$ be a map. Because we have chosen a basis, we have a canonical identification between sub-line bundles $\mathcal{L}' \subset \mathcal{O}_T^{\oplus n+1}$ and quotients $\mathcal{O}_T^{\oplus n+1} \to \mathcal{L}$ – this second interpretation is therefore the same as the choice of $n+1$ globally generating sections of $\mathcal{L}(T)$ (up to $\mathcal{O}_S^\times$). We explain why it is a very good idea to then denote the induced morphism by

$$f \colon T \xrightarrow{[s_0:\cdots:s_n]} \mathbb{P}^n_S.$$

Let $x \in T$ be a point. Then $f(x) \in \mathbb{P}^n_{k(x)}$ which is the set of lines in $k(x)^{\oplus n+1}$. We claim that

$$[s_0(x) \colon \ldots \colon s_n(x)]$$

defines a line in $k(x)^{\oplus n+1}$. Note that by definition $s_i(x) \in \mathcal{L}(x)$ which is not canonically identified with $k(x)$, so we need to explain how to interpret $s_i(x)$ as an element of $k(x)$. Actually we will not claim that this is a well defined element, however we claim that $[s_0(x)\colon \dots : s_n(x)]$ is a well defined line. Because the sections globally generate, $x \in D(s_i)$ for some i say $i = 0$. So we may triviliaze on this open $\frac{1}{s_0}\colon \mathcal{L}_{D(s_0)} \to \mathcal{O}_{D(s_0)}$. Using this trivialization we interpret

$$[s_0(x)\colon \dots : s_n(x)] \quad \text{as} \quad [1 : \frac{s_1}{s_0}(x)\colon \dots : \frac{s_n}{s_0}(x)].$$

This indeed defines a line in $k(x)^{\oplus n+1}$ and one can check that it does not depend on the trivialization. Also note that to understand the image of the dual

$$k(s)^{\oplus n+1} \xrightarrow{(s_0(x)\colon \dots : s_n(x))} \mathcal{L}(x),$$

one can use the above trivialization around x, and then dualize, and the one finds that the line $f(x)$ in question is indeed the one $[s_0(x)\colon \dots s_n(x)]$ whose construction is explained above.

(4) The open complement is given by $D_+(t)$, if t designates the global section $(0,1) \in \mathcal{E} \oplus \mathcal{O}_S$. More precisely, we identify $\mathrm{Sym}(\mathcal{E}^\vee \oplus \mathcal{O}_S) = \mathrm{Sym}(\mathcal{E}^\vee)[t]$. Therefore the degree zero part of the localization by t identifies with $\mathrm{Sym}(\mathcal{E}^\vee)$.

(5) We precise that $\mathbb{V}(\mathcal{O}(1))$ designates the bundle on $\mathbb{P}(\mathcal{E})$.

Note first that the closed subscheme corresponding to $\mathcal{O}_S \subset \mathcal{E} \oplus \mathcal{O}_S$ is given by the graded ideal sheaf $\langle \mathcal{E}^\vee \rangle \subset \mathrm{Sym}(\mathcal{E}^\vee)[t]$. Indeed the above inclusion correspond by duality to the surjection $\mathcal{E}^\vee \oplus \mathcal{O}_S \to \mathcal{O}_S$ and the closed subscheme corresponding is given by the graded ideal sheaf generated by the kernel of this map.

Note that the the graded inclusion $\mathrm{Sym}(\mathcal{E}^\vee) \to \mathrm{Sym}(\mathcal{E}^\vee)[t]$ induces by functoriality of Proj a morphism $U \to \mathbb{P}(\mathcal{E})$ where U is the open subscheme complement of the closed subscheme mentioned above.

Our goal is now to show that $U \to \mathbb{P}(\mathcal{E})$ is isomorphic as $\mathbb{P}(\mathcal{E})$-scheme to $\mathbb{V}(\mathcal{O}(1)) \to \mathbb{P}(\mathcal{E})$. To this end we view these as functors on $\mathbb{P}(\mathcal{E})$-schemes *via* the Yoneda embedding. It suffices therefore to construct a natural isomorphism of their respective functor of points. Recall that a $\mathbb{P}(\mathcal{E})$-scheme is the data of a morphism $g\colon T \to \mathbb{P}(\mathcal{E})$ and therefore the data of a sub-line bundle $\mathcal{L}_T \subset f^*\mathcal{E}$, if $f\colon T \to S$ denotes the composition of g with the projection to S. When we write T in what follows, we carry implicitly the above information. The points of U are then

$$U(T) = \{\mathcal{M} \subset f^*\mathcal{E} \oplus \mathcal{O}_T \mid \mathcal{M} \text{ a sub-l.b. and } \mathcal{M}_{|\mathcal{E}} = \mathcal{L}_T\}.$$

The points of $\mathbb{V}(\mathcal{O}(1))$ are

$$\mathbb{V}(\mathcal{O}(1))(T) = \mathcal{L}_T^\vee(T).$$

To see that these two functors are isomorphic, note that sending $\mathcal{M} \in U(T)$ to

$$\phi_{\mathcal{M}} \colon \mathcal{L} \subset \mathcal{M} \subset f^*\mathcal{E} \oplus \mathcal{O}_T \to \mathcal{O}_T$$

and $\phi \in \mathcal{L}_T^\vee(T)$ to the sub-line bundle generated by

$$\langle (v, \phi(v)) \rangle_{v \in \mathcal{L}_T} \in f^*\mathcal{E} \oplus \mathcal{O}_T$$

is an isomorphism.

□

Exercise 5. *Tautological line bundle.* This exercise is a direct follow-up to the preceding one. We call

$$\mathcal{O}(-1) \subset \pi^*\mathcal{E}$$

the *tautological line bundle.* We gather in this exercise various properties of this *universal* line bundle.

Say U is an affine of S, $M = \mathcal{E}(U)$ and $\varphi \in M^\vee$. Let $c \in M \otimes M^\vee$ be the canonical element (corresponding to the identity along the natural isomorphism $M \otimes M^\vee \cong \operatorname{Hom}_A(M, M)$).

(1) Show that $\mathcal{O}(-1)$ can be realized as the sub-line bundle of $\pi^*\mathcal{E}$ generated on $D_+(\varphi)$ by

$$c/\varphi \in \pi^* M(D_+(\varphi)) = \operatorname{Sym}(M^\vee)_{(\varphi)} \otimes M.$$

(2) Let $f \colon T \to S$ an S-scheme. From the previous exercise, deduce that if $T \to \mathbb{P}(\mathcal{E})$ is the map of S-schemes corresponding to an $\mathcal{L} \subset \pi^*\mathcal{E}$, then the following square

$$\begin{array}{ccc} \mathbb{V}(\mathcal{L}) & \longrightarrow & \mathbb{V}(\mathcal{O}(-1)) \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathbb{P}(\mathcal{E}) \end{array}$$

is Cartesian.

Remark. The above says that $\mathbb{P}(\mathcal{E})$ is the *moduli space of sub-line bundles of* $\mathcal{E}$, and that $\mathcal{O}(-1)$ is the *universal line bundle on the moduli.*

(3) Show that $\mathbb{V}(\mathcal{O}(-1))$ is a closed subscheme of $\mathbb{V}(\pi^*\mathcal{E}) = \mathbb{V}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E})$. This comes from the surjection $\pi^*\mathcal{E}^\vee \to \mathcal{O}(1)$.

(4) Let $f \colon T \to S$ an S-scheme. Show that a map of S-schemes $T \to \mathbb{V}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E})$ which corresponds to a pair $(\mathcal{L}, v)$ with $\mathcal{L} \subset f^*\mathcal{E}$ and $v \in f^*\mathcal{E}(T)$ factors through $\mathbb{V}(\mathcal{O}(-1))$ if and only if $v \in \mathcal{L}(T)$.

Remark. In particular if $S = \operatorname{Spec}(k)$ where k is a field, and $\mathcal{E} = k^{n+1}$, the bundle $\mathbb{V}(\mathcal{O}(-1))$ is realized as a closed subscheme of $\mathbb{A}_k^{n+1} \times_k \mathbb{P}_k^n$.

Solution key. (1) A local claim suffices. So say $S = \operatorname{Spec}(A)$ is affine and $\mathcal{E}$ can be identified with a finite projective A-module M. The built-in surjection $\pi^* M^\vee \to \mathcal{O}(1)$ reads on $D_+(\varphi)$ as the surjective map

$$\alpha \colon \operatorname{Sym}(M^\vee)_{(\varphi)} \otimes M^\vee \to \mathcal{O}(1)(D_+(\varphi))$$

determined by $1 \otimes \psi \mapsto \psi$. We want to dualize it to understand the claim. Namely we want to understand the image of the dual. Note that to determine that, we can post compose by an isomorphism the above map, so we can trivialize by $\frac{1}{\varphi}\colon \mathcal{O}(1)(D_+(\varphi)) \to \mathcal{O}(D_+(\phi))$. Therefore we want to analyze the dual of the map

$$\mathrm{Sym}(M^\vee)_{(\varphi)} \otimes M^\vee \to \mathrm{Sym}(M^\vee)_{(\varphi)}$$

determined by sending $1 \otimes \psi \to \frac{\psi}{\phi}$.

Recall that (because M is finite projective) we have an isomorphism

$$\mathrm{Sym}(M^\vee)_{(\varphi)} \otimes M \to \mathrm{Hom}_{\mathrm{Sym}(M^\vee)_{(\varphi)}}(\mathrm{Sym}(M^\vee)_{(\varphi)} \otimes M^\vee, \mathrm{Sym}(M^\vee)_{(\varphi)})$$

via the map determined by sending $m \in M$ to the map determined by sending $\psi \in M^\vee$ to $\psi(m)$. The image of the dual of α on the right side is given, as explained above by the map determined by $1 \otimes \psi \mapsto \frac{\psi}{\varphi}$. So it suffice to check that $\frac{c}{\varphi}$ is sent to this map. Recall that if $c = \sum_i \phi_i \otimes m_i$, it has the property that for every $\psi \in M^\vee$ we have

$$\psi = \sum_i \psi(m_i)\phi_i.$$

Therefore the claim follows.

(2) By construction in the situation of the previous exercise $f^*\mathcal{O}(-1) = \mathcal{L}$. The claim follows.

(4) For the two last items, one notes that translating the factorization into the closed subscheme $\mathbb{V}(\mathcal{O}(-1))$ amounts to the existence of a factorization

$$\begin{array}{ccc} f^*\mathcal{E}^\vee & & \\ \downarrow & \searrow & \\ \mathcal{L}^\vee & \dashrightarrow & \mathcal{O}_T \end{array}$$

which amounts by dualizing to the claim.

Remark. Yet another perspective on $\mathbb{V}(\mathcal{O}(-1))$ is that it is the *blow-up of* $\mathbb{V}(\mathcal{E})$ *at the zero section.*

□

Exercise 6. *Stability properties of (very-)ample sheaves under tensor product.* Let X be a Noetherian scheme. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on X.

(1) If $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, show that $\mathcal{L} \otimes \mathcal{M}$ is ample.
(2) If $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, deduce that there is a n such that $\mathcal{L}^n \otimes \mathcal{M}$ is ample.
(3) Show that if $\mathcal{L}$ and $\mathcal{M}$ are ample, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

Now suppose that X is an A-scheme where A is a Noetherian ring.

(4) If $\mathcal{L}$ is A-very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is A-very ample.

(5) If $\mathcal{L}$ is ample, then there is a $n_0 > 0$ such that $\mathcal{L}^n$ is A-very-ample for all $n \geq n_0$.

Solution key. (1) First note that if $\mathcal{F}$ and $\mathcal{G}$ are globally generated and then $\mathcal{F} \otimes \mathcal{G}$ is also because all pure tensors of global sections $f \otimes g$ because this is a local claim and the tensor product of two surjective map is surjective. The claim follows.

(2) Follows.

(3) Same.

(4) Choose sections $s_0, \ldots, s_n \in \mathcal{L}(X)$ which defines an A-immersion $X \to \mathbb{P}^n_A$ and $m_0, \ldots, m_m \in \mathcal{M}(X)$ that defines an A-morphism $X \to \mathbb{P}^m_A$. Then the product $X \to \mathbb{P}^n_A \times_A \mathbb{P}^m_A$ is an immersion as immersions are closed under base-change. Now conclude using a Segre embedding.

(5) Follows from previous point and the proposition shown in class that there exists *some* n_0 with $\mathcal{L}^{n_0}$ being A-very ample.

□