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AG II - Schemes and sheaves

Exercises – week 11

Exercise 1. *Tensor products,* Hom *and sheafification.* Give examples of sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ such that the tensor product presheaf and the presheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$$

are not sheaves.
Hint: Play with $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ on projective spaces. Recall the computation of global sections of those, exercise 5, week 10.

Exercise 2. *Effective Cartier divisors.* Let X be an integral scheme. A Cartier divisor on X represented by (f_i, U_i) is said to be *effective* if $f_i \in \mathcal{O}(U_i)$ for every i.

(1) Show, by looking at the ideal sheaf generated by the f_i's, that effective Cartier divisors are in one-to-one correspondence with ideal sheaves $\mathcal{I}$ that are a locally free sheaves of rank 1. We take this point of view in what follows.
(2) Let $\mathcal{L}$ be a locally free sheaf of rank 1. Show that $s\colon \mathcal{O}_X \to \mathcal{L}$ is non-zero if and only if the evaluation $\mathrm{ev}_s : \mathcal{L}^\vee \to \mathcal{O}_X$, defined by $\mathcal{L}^\vee(U) = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{L}_U, \mathcal{O}_u) \ni \varphi \mapsto \varphi(s)$ is injective.
(3) Fix a locally free sheaf $\mathcal{L}$ of rank 1. Deduce the following bijection,

$$\left(\Gamma(X, \mathcal{L}) \setminus \{0\}\right) / \mathcal{O}_X(X)^\times \to \{\text{Effective Cartier divisors } \mathcal{I} \text{ on } X \text{ with } \mathcal{I} \cong \mathcal{L}^\vee\}$$

that sends the class of a section s to $\mathrm{Im}(\mathrm{ev}_s)$.
(4) Suppose that $\mathcal{O}_X(X)$ is a field. Show that if $\mathcal{L}$ is a locally free sheaf of rank 1 such that $\mathcal{L}$ and $\mathcal{L}^\vee$ have a non zero section, then $\mathcal{L} \cong \mathcal{O}_X$.
Hint: in this case both $\mathcal{L}$ and $\mathcal{L}^\vee$ correspond to effective Cartier divisors.
(5) Additionally assume that X is normal, Noetherian and integral. *Two Weil divisors are called linearly equivalent if their difference is the divisor of some rational function.* Let D be a Weil divisor on X. Show that map sending $f \in \Gamma(X, \mathcal{O}_X(D))$ to $\mathrm{div}(f) + D$ gives a one to one correspondence

$$\frac{\Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}}{\mathcal{O}_X(X)^\times} \to \{\text{Effective Weil divisors linearly equivalent to } D\}.$$

Careful, hypothesis does not imply that $\mathcal{O}_X(D)$ defined as

$$\mathcal{O}_X(D)(U) = \{g \in K(X) \mid g \neq 0, (div(g) + D) \cap U \text{ is effective}\}$$

is a line bundle. So you have to prove it independently of item (3).

Exercise 3. *Invertible sheaves and cocycles, a first encounter.* Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}$ be an invertible sheaf on X. Let (U_i) be a cover of X with trivializations $\varphi_i \colon \mathcal{L}_{U_i} \to \mathcal{O}_{U_i}$. We say that the associated cocycles $(\varphi \in \mathcal{O}_X(U_{ij})^\times)$ are defined to be $\varphi_i \circ \varphi_j^{-1} \colon \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$ that we identify with $\varphi_{ij} \in \mathcal{O}_X(U_{ij})^\times$. Say $\mathcal{L}'$ is another invertible sheaf with associated cocycles (ψ_{ij}).

(1) Show that $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ and $\varphi_{ii} = 1$.
(2) Show that cocycles associated to $\mathcal{L}^\vee$ are (φ_{ij}^{-1}), and cocycles associated to $\mathcal{L} \otimes \mathcal{L}'$ are $(\varphi_{ij}\psi_{ij})$.
(3) Show that if for every i there is some $h_i \in \mathcal{O}(U_i)^\times$ such that $h_i \varphi_{ij} h_j^{-1} = \psi_{ij}$, then $\mathcal{L} \cong \mathcal{L}'$.

We will go further in this study when introducing *first Čech cohomology.*

Exercise 4. *Extension of coherent sheaves.* The goal is to show that if X is a Noetherian scheme, U an open subset and $\mathcal{F}$ is a coherent sheaf on U, then there is a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}_{|U} \cong \mathcal{F}$.

(1) Show that on a Noetherian scheme X and $\mathcal{F}$ coherent sheaf, then if
$$\sum_i \mathcal{F}_i = \mathcal{F}$$
where $(\mathcal{F}_i)_{i \in I}$ are sub-coherent sheaves, then there exist a finite refinement $J \subset I$ such that $\sum_{j \in J} \mathcal{F}_j = \mathcal{F}$.
(2) Show that on a Noetherian affine scheme, every quasi-coherent sheaf is the direct colimit of it's coherent sub-sheaves. *Hint: Use the equivalence of categories with modules on global sections.*
(3) Let X be affine and $\iota \colon U \to X$ be an open subcheme. Show the claim in this case. *Hint: Show that $\iota_* \mathcal{F}$ is quasi-coherent, and then use a combination of (1) and (2) to conclude.*
(4) Show the claim in the general case of the statement of the exercise by induction on the number of open affines that are required to cover X. (Being covered by one open affine being the base case of the induction, and is the previous point. The rest is an induction play, see Hint.) *Hint: Say $X = X_1 \cup X_2$ where X_1 and X_2 are open subschemes that can be covered by strictly less open affines than X. By induction extend $\mathcal{F}_{X_1 \cap U}$ to a coherent sheaf $\mathcal{G}_1$ defined on X_1. By gluing $\mathcal{F}$ and $\mathcal{G}_1$ it defines a coherent sheaf $\mathcal{G}'$ defined on $X_1 \cup U$. Now, extend $\mathcal{G}'_{|X_2 \cap (X_1 \cup U)}$ to a coherent sheaf $\mathcal{G}_2$ on X_2. Conclude by gluing $\mathcal{G}_1$ and $\mathcal{G}_2$ to a coherent sheaf on X.*

As an application, show that any quasi-coherent sheaf on a Noetherian scheme is a direct colimit of sub-coherent sheaves.

Exercise 5. *Divisors on regular curves.* Let k be an algebraically closed field. We say that C is a *regular k-curve over k* is a one dimensional separated, integral and regular scheme over k. Weil (=Cartier in this case) divisors are then of the form

$$D = \sum_i n_i x_i$$

for x_i being closed points of C. We define the *degree* of a divisor $D = \sum_i n_i x_i$ to be

$$\deg(D) = \sum_i n_i \in \mathbb{Z}.$$

Let $f\colon C' \to C$ a finite k-morphism between regular k-curves. We define the *pullback* of an irreducible divisor (=closed point)

$$f^* x = \sum_{y \in C'_{cl} \text{ s.t. } x = f(y)} v_y(f^\sharp(t_x)) y.$$

where $f^\sharp$ denotes the induced map at the local ring. Here, t_x denotes a generator of $\mathfrak{m}_x$ – this well defined because the choice of a generator is up to a unit. We extend f^* by linearity to $\mathrm{Div}(C)$.

(1) Show that the pullback of a principal divisor is principal, implying that f^* factors through

$$f^*\colon\ \mathrm{Cl}(C) \to \mathrm{Cl}(C').$$

(2) Show that if the degree of the map ($= [K(C')\colon K(C)]$) is d, then $\deg(f^* D) = d \deg(D)$. *Hint: it suffices to show the claim for $D = x$ a closed point by linearity.*

(3) Assume now that C is also proper. Using the equivalence of categories seen in lecture on curves (that you can assume) between k-fields of k-transcendence degree 1 and regular proper k-curves, show that for every $t \in K(C) \backslash k$ we have a map $f_t\colon C \to \mathbb{P}^1_k$ from the inclusion $k(t) \subset K(C)$ such that $f^*(0 - \infty) = (f)$ where 0 denotes $V(f)$ in $\mathrm{Spec}(k[f]) \subset \mathbb{P}^1_k$ and ∞ denotes $V(1/f)$ in $\mathrm{Spec}(k[1/f]) \subset \mathbb{P}^1_k$. deduce that $\deg((f)) = 0$, and that therefore deg factor through

$$\deg\colon\ \mathrm{Cl}(C) \to \mathbb{Z}.$$

Exercise 6. *Segre viewed with line bundles.* Fix an algebraically closed field k. Denote the projection of $\mathbb{P}^1 \times_k \mathbb{P}^1$ to the first and second factor by p_1 and p_2 respectively. View the first and second copy of $\mathbb{P}^1$ in the product as $\mathrm{Proj}(k[x_0, x_1])$ and $\mathrm{Proj}(k[y_0, y_1])$ respectively. Show that the global sections $p_1^*(x_i) \otimes p_2^*(y_j)$ for $0 \le i, j \le 1$ of $p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$ give a closed embedding of $\mathbb{P}^1 \times_k \mathbb{P}^1$ in $\mathbb{P}^3$.