Series 7 - November 6, 2024

Exercise 1.

Consider the following SDE

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dW(t), \quad t \in [0,T] \\ X(0) &= X_0, \end{aligned} \tag{1.1}$$

with $X(0) \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma > 0$. Equation (1.1) admits a unique closed-form solution and it is called Geometric Brownian motion.

1) Find the closed form solution for (1.1).
2) Consider $X(0) = 1$, $\mu = 2$, $\sigma = 1$, and $T = 1$. Compute the Euler-Maruyama discretization $\{X_n\}_{n=0}^N$ of (1.1) for $\Delta t = \frac{T}{n} = 0.1, 0.05, 0.01, 0.005, 0.001$ and for a single realization plot $\sup_{0 \leqslant n \leqslant N} |X_{t_n} - X_n|$ versus $(\Delta t)^{\frac{1}{2}}$.
3) Compute $Z_N = \sup_{0 \leqslant n \leqslant N} \frac{|X_{t_n} - X_n|}{(\Delta t)^{\frac{1}{2}}}$ for $\Delta t = T/N$ with $N = 100$ and $M = 1000$ independent trajectories and plot the empirical CDF(Z_N). Do the same computation for $N = 1000$; does CDF(Z_N) converge to a limit distribution ?
4) What happens if you take $Z_N^\alpha = \sup_{0 \leqslant n \leqslant N} \frac{|X_{t_n} - X_n|}{(\Delta t)^\alpha}$ for $\alpha < \frac{1}{2}$? Is this result consistent with your expectations ?

Exercise 2.

Consider the $1D$ autonomous SDE

$$\begin{cases} dX_t &= b(X_t)dt + \sigma(X_t)dW_t, \quad t \in [0,T], \\ X(0) &= \eta \in L^2(\Omega), \end{cases} \tag{2.1}$$

and the stochastic θ-method

$$X_{n+1} = X_n + \theta \Delta t b(X_{n+1}) + (1-\theta)\Delta t b(X_n) + \sigma(X_n)\Delta W_n \tag{2.2}$$

Assume b, σ to satisfy a global Lipschitz condition and a linear growth bound and, moreover, b is continuously differentiable with bounded derivative $|b'(x)| \leqslant K$ for all $x \in \mathbb{R}$.

1) Show that if $\Delta t < \frac{1}{K\theta}$ the numerical solution is well defined (uniquely exists)

   Hint. You can show that the map $\varphi(x) = \theta b(x)\Delta t + \gamma$, $\gamma = X_n + (1-\theta)b(X_n)\Delta t + \sigma(X_n)\Delta W_n$ is contractive
2) Show that $\exists C > 0 \ : \ \sup_{0 \leqslant n \leqslant N} \mathbb{E}[|X_n|^2] \leqslant C$

   Hint. multiply (2.2) by X_n and use the identity $(X_{n+1} - X_n)X_n = \frac{1}{2}X_{n+1}^2 - \frac{1}{2}X_n^2 - \frac{1}{2}(X_{n+1} - X_n)^2$

3) Defining the (non adapted) process

$$\begin{aligned}\hat{X}_s &= X_n + (\theta b(X_{n+1}) + (1-\theta)b(X_n))(s-t_n) + \sigma(X_n)(W_s - W_{t_n}) \\ &= X_n + \int_{t_n}^{s} (\theta b(X_{n+1}) + (1-\theta)b(X_n))d\tau + \int_{t_n}^{s} \sigma(X_n)dW_\tau, \quad t_n < s \leqslant t_{n+1}.\end{aligned} \tag{2.3}$$

Redo the same steps of the proof of the strong convergence of Euler-Maruyama to show that $\mathbb{E}[\sup_{0\leqslant t\leqslant T} |X_s - \hat{X}_s|^2] \leqslant C\Delta t$.

Exercise 3.

Let $b > 0$, $\sigma \in \mathbb{R}$ and $X_0 \in \mathbb{R}$ and consider the Langevin equation

$$\begin{aligned}&\mathrm{d}X(t) = -bX(t)\mathrm{d}t + \sigma \mathrm{d}W(t), \quad t \in [0,T], \\ &X(0) = X_0.\end{aligned} \tag{3.1}$$

Remark. The solution $X(t)$ is also called the Ornstein–Uhlenbeck process.

i) Solve equation (3.1).

ii) Verify that $\lim_{t\to\infty} \mathbb{E}[X(t)] = 0$ and $\lim_{t\to\infty} \mathrm{Var}(X(t)) = \frac{\sigma^2}{2b}$ and the distribution of the limit random variable $X(\infty)$ is $\mathcal{N}(0, \frac{\sigma^2}{2b})$.

Remark. The limit distribution is still $\mathcal{N}(0, \frac{\sigma^2}{2b})$ if X_0 is a Gaussian random variable independent of the Brownian motion.

The stochastic θ-method applied to the SDE

$$\begin{aligned}&\mathrm{d}X(t) = f(t, X(t))\mathrm{d}t + g(t, X(t))\mathrm{d}W(t), \\ &X(0) = X_0,\end{aligned}$$

is defined for $\theta \in [0,1]$ and a partition $P = \{0 = t_0 < t_1 < ... < t_N = T\}$ of size Δt as

$$X_{n+1} = X_n + f(t_n, X_n)\Delta t(1-\theta) + f(t_{n+1}, X_{n+1})\Delta t\theta + g(t_n, X_n)(W(t_{n+1}) - W(t_n)).$$

For a given $\epsilon > 0$, set $T = 1$, $\sigma = \sqrt{2/\epsilon}$, $b = 1/\epsilon$ and $X_0 = 1$ and apply the θ-method to approximate the solution $X(t)$ of (3.1).

iii) Set $\epsilon = 1/20$. Approximate the solution of equation (3.1) employing the θ-method with $\theta = 0, 1/2, 1$ and uniform partitions $P_k = \{0 = t_0 < t_1 < ... < t_{N_k} = 1\}$ with $N_k = 2^k$ and $k = 2, 4, 6$. Verify that the θ-method with $\theta = 0$ is unstable for large values of Δt.

iv) Consider a uniform partition $P = \{0 = t_0 < t_1 < ... < t_N = 1\}$ with $N = 2^6$. For $\epsilon = 1/20, 1/40, 1/60$ approximate the probability density function f of X_N employing the θ-method with $\theta = 0, 1/2, 1$. Verify that the θ-method with $\theta = 1/2$ is the only one which preserves the limit distribution $N(0, \frac{\sigma^2}{2b})$.

Hint. In order to approximate the density function f of X_N, make a histogram of X_N for $M = 10^4$ independent sample paths and normalize it so that $\int_{\mathbb{R}} f\mathrm{d}x = 1$.

Exercise 4.

Consider the SDE

$$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \tag{4.1}$$

with initial condition $X(0) = X_0$ and where W is a one-dimensional Brownian motion. Show that the fully implicit method

$$X_{n+1} = X_n + \lambda X_{n+1} \Delta t + \mu X_{n+1} \Delta W_n,$$

has unbounded first moments, i.e., $\mathbb{E}[|X_n|] = +\infty$ for all n.

Exercise 5.

Show that the stochastic Heun method

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{1}{2}(b(Y_n, t_n) + b(\bar{Y}_n, t_n))\Delta t + \frac{1}{2}(\sigma(Y_n, t_n) + \sigma(\bar{Y}_n, t_n))\Delta W_n \\ \bar{Y}_n &= Y_n + b(Y_n, t_n)\Delta t + \sigma(Y_n, t_n)\Delta W_n \end{aligned}$$

with fixed time step Δt, i.e. $t_n = n\Delta t$ for all n, is not consistent when applied to the SDE $dX_t = 2X_t dW_t$, $t > 0$, $X_0 = 1$.

Hint: show that $\mathbb{E}[X_t] = 1$, $\forall t$, whereas $\mathbb{E}[Y_n] \not\to 1$ as $\Delta t \to 0$.