Series 2 - September 25, 2024

Exercise 1.

Let $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geqslant 0}\right)$ be a stochastic basis. Consider two stochastic processes $X, Y : \Omega \to \mathbb{R}^T$, with T an interval of $\mathbb{R}^+$ or the whole $\mathbb{R}^+$, both adapted to $(\mathcal{F}_t)_{t\geqslant 0}$. Show that if X, Y are a modification of each other and they are a.s. continuous, then they are indistinguishable

Exercise 2.

Let $(X(t), 0 \leqslant t \leqslant T)$ and $(Y(t), 0 \leqslant t \leqslant T)$ be two stochastic processes. By providing counterexamples, show that:

i) $P\big(X(t) = Y(t)\ \forall\, t \in \mathbb{Q} \cap [0,T]\big) = 1 \quad \not\Longrightarrow \quad P\big(X(t) = Y(t)\ \forall\, t \in [0,T]\big) = 1.$

ii) $P\big(X(t) = Y(t)\big) = 1\ \forall\, t \in [0,T] \quad \not\Longrightarrow \quad P\big(X(t) = Y(t)\ \forall\, t \in [0,T]\big) = 1.$

(In view of the previous exercise, X, Y cannot be both a.s. continuous).

Exercise 3.

Let B be a real Brownian motion and $\pi = \{t_0, \dots, t_m\}$ with $0 \leqslant s = t_0 < t_1 \cdots < t_m = t$ be a partition of the interval $[s,t]$, with $|\pi| = \max_{0\leqslant k\leqslant m-1} |t_{k+1} - t_k|$.

1) Show that
$$V_\pi^2 = \sum_{k=0}^{m-1} \left|B_{t_{k+1}} - B_{t_k}\right|^2$$
satisfies
$$\lim_{|\pi|\to 0^+} V_\pi^2 = t - s \quad \text{in } L^2.$$

2) Show that $\lim\limits_{|\pi|\to 0^+} V_\pi^1 = \infty$ a.s. $\forall t > s$, i.e. the paths of a Brownian motion do not have finite variation in any time interval a.s.

Exercise 4.

Consider a Brownian motion B on a filtered probability space $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geqslant 0}\right)$. Show that

1) for every $0 \leqslant s < t$ the r.v. $B_t - B_s$ is independent of B_u, $\forall u \leqslant s$;

2) B is a Gaussian process.

Hint. To show that a stochastic processes $\{X_t\}_{t\in[0,\infty)}$ is a Gaussian it is enough to show that for any $t_1, t_2, \dots, t_m \in \mathbb{R}^+$ and any $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}$ the random variable $Z = \sum_{i=1}^m \alpha_i X_{t_i}$ is Gaussian.

Exercise 5.

The family of Haar functions $\{h_k\}_{k\geqslant 0}$ is defined for $0 \leqslant t \leqslant 1$ as

$$h_0(t) = 1, \qquad h_1(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant 1/2, \\ -1 & \text{if } 1/2 < t \leqslant 1, \end{cases}$$

and for $2^n \leqslant k < 2^{n+1}$ with $n = 1, 2, \ldots$ as

$$h_k(t) = \begin{cases} 2^{n/2} & \text{if } \frac{k-2^n}{2^n} \leqslant t \leqslant \frac{k-2^n+1/2}{2^n}, \\ -2^{n/2} & \text{if } \frac{k-2^n+1/2}{2^n} < t \leqslant \frac{k-2^n+1}{2^n}, \\ 0 & \text{otherwise.} \end{cases} \qquad n = \lfloor \log_2 k \rfloor$$

i) Show that $\{h_k\}_{k\geqslant 0}$ is orthonormal in $\mathrm{L}^2(0,1)$.

ii) Show that $\{h_k\}_{k\geqslant 0}$ is complete in $\mathrm{L}^2(0,1)$, i.e., $f = \sum_{k=0}^{\infty} \langle f, h_k \rangle h_k$ in $\mathrm{L}^2(0,1)$ for any $f \in \mathrm{L}^2(0,1)$.
Hint. First prove that if $\langle g, h_k \rangle = 0$ for all $k \geqslant 0$ then $g = 0$ a.s. by showing that $\int_s^t g = 0$ for all $0 \leqslant s \leqslant t \leqslant 1$.

Exercise 6.

The family of Schauder functions $\{s_k\}_{k\geqslant 0}$ is defined for $0 \leqslant t \leqslant 1$ as $s_k(t) = \langle \chi_{[0,t]}, h_k \rangle$, where $\{h_k\}_{k\geqslant 0}$ are the Haar functions and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathrm{L}^2(0,1)$. Then let $W(t) = \sum_{k=0}^{\infty} \xi_k s_k(t)$, where $\{\xi_k\}_{k\geqslant 0}$ is a sequence of independent standard Gaussian random variables $\xi_k \sim N(0,1)$.

i) Show that $\sum_{k=0}^{\infty} s_k(r) s_k(t) = \min\{r, t\}$ for all $0 \leqslant r, t \leqslant 1$.

ii) Show that there exists a constant $C > 0$ such that for all $k \geqslant 2$ it holds

$$P(|\xi_k| > 4\sqrt{\log k}) \leqslant C k^{-4},$$

and deduce that almost surely there exists a positive integer $\bar{k}$ such that for all $k > \bar{k}$ it holds $|\xi_k| \leqslant 4\sqrt{\log k}$.
Hint. Apply Borel–Cantelli lemma.

iii) Prove that the series $W(t)$ converges uniformly for $t \in [0,1]$.
Hint. You can follow these steps where $C > 0$ is a positive constant.

a) Show that almost surely for n big enough $\max_{2^n \leqslant k < 2^{n+1}} |\xi_k| \leqslant C 2^{\frac{n+1}{4}}$.

b) Show that $\sum_{k=2^n}^{2^{n+1}-1} |s_k(t)| \leqslant 2^{-\frac{n+2}{2}}$.

c) Show that for m big enough $\sum_{k=2^m}^{\infty} |\xi_k||s_k(t)| \leqslant C \sum_{n=m}^{\infty} 2^{-\frac{n+3}{4}}$.

Exercise 7.

The family of Schauder functions $\{s_k\}_{k\geqslant 0}$ is defined for $0 \leqslant t \leqslant 1$ as $s_k(t) = \langle \chi_{[0,t]}, h_k \rangle$, where $\{h_k\}_{k\geqslant 0}$ are the Haar functions and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathrm{L}^2(0,1)$. Then let

$$W(t) = \sum_{k=0}^{\infty} \xi_k s_k(t), \tag{7.1}$$

where $\{\xi_k\}_{k\geqslant 0}$ is a sequence of independent standard Gaussian random variables $\xi_k \sim N(0,1)$.

Show that $W(t)$ is actually a Brownian motion. Equation (7.1) is the Lévy-Ciesielski construction of the Brownian motion.