Recall: Characters of representations.

Def. Let V be a representation of an algebra A. Then the character $\chi_V : A \to k$ is given by

$$\chi_V(a) = \operatorname{Tr}_V \rho(a)$$

Today: Characters of finite groups. Orthogonality of characters.

Theorem. (1) Characters of distinct irreducible finite dimensional representations of A are linearly independent.

(2) If A is a finite dimensional semisimple algebra then these characters $\{\chi_{V_i}\}_{i=1}^r$ form a basis in $\left(A/[A,A]\right)^*$.

Proof: (1) last time

(2) $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] = sl_d(k)$:= traceless matrices $d \times d$.

Clearly $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] \subset sl_d(k)$

$tr(xy - yx) = 0$

We have $\left.\begin{array}{l} [E_{ij}, E_{jm}] = E_{im} \\ i \neq m \quad [E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1} \end{array}\right\}$ form a basis in $sl_d(k)$

$\Rightarrow [\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] = sl_d(k)$

Since A is semisimple $\Rightarrow A \simeq \mathrm{Mat}_{d_1}(k) \oplus \dots \oplus \mathrm{Mat}_{d_r}(k)$

Then $[A, A] = \underbrace{sl_{d_1}(k)}_{d_1^2 - 1} \oplus \dots \oplus \underbrace{sl_{d_r}(k)}_{d_r^2 - 1} \Rightarrow \dim A/[A,A] = r$

$\exists !$ irreducible representation of A for each component $\mathrm{Mat}_{d_i}(k)$, up to isomorphism. $\Rightarrow \exists$ exactly r irreducible representations of A.

$\Rightarrow$ We have r linearly independent characters $\Rightarrow$ they form a basis in $\left(A/[A,A]\right)^*$.

$\blacksquare$

Example. of the decomposition of a group algebra into a direct sum of matrix algebras.

Let $A = \mathbb{C}[C_2]$ $\quad C_2 = \{1, s\}$

V_0 trivial $\rho_0(1) = \rho_0(s) = 1$; V_s sign : $\rho_s(1) = 1, \rho_s(s) = -1$.

$A = Mat_1(\mathbb{C}) \oplus Mat_1(\mathbb{C}) = \mathbb{C} e_1 + \mathbb{C} e_2$, where $e_1 = \frac{(1+s)}{2}$, $e_2 = \frac{(1-s)}{2}$

$\Rightarrow e_1^2 = \frac{1}{4}(1+2s+1) = \frac{1+s}{2} = e_1$, $e_2^2 = \frac{1}{4}(1-2s+1) = \frac{1-s}{2} = e_2$; $e_1 e_2 = \frac{1}{4}(1-s^2) = 0$,

$e_1 + e_2 = \frac{1}{2}(1+s) + \frac{1}{2}(1-s) = 1$. $\Rightarrow \{e_1, e_2\}$ are orthogonal idempotents in $\mathbb{C}[C_2]$

Left regular representation of A in this basis:

$s \cdot e_1 = s \cdot \frac{1}{2}(1+s) = \frac{1}{2}(1+s)$ $\qquad s \cdot e_2 = s \cdot \frac{1}{2}(1-s) = \frac{1}{2}(s-1) = -\frac{1}{2}(1-s)$

$\Rightarrow \mathbb{C} e_1 \simeq V_0$ $\qquad \Rightarrow \mathbb{C} e_2 \simeq V_s$

$\Rightarrow A = V_0 \oplus V_s$ as the left regular representation.

Characters of finite groups.

Def. If G is a finite group, V a finite dimensional representation. Then $\chi_V : G \to k$ is defined as $\chi_V(g) = Tr_V(\rho(g))$

Clearly V is also a representation of $k[G]$ and $\chi_V(g)$ carries the same information as $\chi_V(a)$, $a \in k[G]$

Claim 1. $\chi_V : G \to k$ is a class function: (it only depends on the conjugacy class of the group element).

$$\chi_V(hgh^{-1}) = Tr_V\, \rho(hgh^{-1}) = Tr_V\, \rho(h)\rho(g)(\rho(h))^{-1} = Tr_V\, \rho(g) = \chi_V(g)$$

cyclicity of trace.

Def. Let $F(G,k)$ be the space of k-valued functions on G and $F_c(G,k) \subset F(G,k)$ the subspace of class functions

Theorem The characters of irreducible representations of G over $\mathbb{C}$ form a basis in the space $F_c(G,\mathbb{C})$.

Proof. $\mathbb{C}[G]$ is semisimple $\Rightarrow$ the characters of irreducible representations form a basis in $\left(A/[A,A]\right)^*$ where $A = \mathbb{C}[G]$.

$$\left(A/[A,A]\right)^* = \{\varphi \in Hom_{\mathbb{C}}(\mathbb{C}[G], \mathbb{C}) : xy - yx \in \ker\varphi \ \forall x,y \in \mathbb{C}[G]\}$$

$$\Leftrightarrow gh - hg \in \ker\varphi \ \forall g,h \in G$$

$$\Leftrightarrow \{f \in F(G,\mathbb{C}) : f(gh) = f(hg) \ \forall g,h \in G\}$$

$$\iff \{ f \in F(G,\mathbb{C}) : f(hgh^{-1}) = f(g) \quad \forall g,h \in G \}$$

$f(hgh^{-1}) = f(h^{-1}hg) = f(g)$; $f(\underbrace{hgh^{-1}}_{k}) = f(g) \Rightarrow f(hk) = f(kh)$.

$k = gh^{-1} \Rightarrow g = kh$

$$= F_c(G,\mathbb{C}).$$

$\blacksquare$

Corollary 1 The number of isomorphism classes of irreducible representations of G over $\mathbb{C}$ is equal to the number of conjugacy classes in G.

Corollary 2. Any representation of G over $\mathbb{C}$ is completely determined by its character. : $\chi_V = \chi_W \iff V \simeq W$

Proof: $V = \bigoplus_{i=1}^{r} V_i^{\oplus n_i}$ $\quad W \simeq \bigoplus_{i=1}^{r} V_i^{\oplus m_i}$

$\{\chi_i\}_{i=1}^{r}$ is a basis

$\chi_V = \sum_{i=1}^{r} n_i \chi_i$ $\quad \chi_W = \sum_{i=1}^{r} m_i \chi_i$

decomposition is unique

Conclusions.

(1) $\mathbb{C}[G]$ is semisimple and $\mathbb{C}[G] \simeq \bigoplus_{i=1}^{r} \mathrm{Mat}_{n_i}(\mathbb{C})$

where $\{V_i\}_{i=1}^{r}$ with $\dim V_i = n_i$ are the irreducible representations of G.

(2) $|G| = \sum_{i=1}^{r} n_i^2$ where $n_i = \dim V_i$.

Example. Recall the irreducible representations of D_4:

V_0: $\rho(s)=1, \rho(r)=1$

V_1: $\rho(s)=-1, \rho(r)=1$

V_2: $\rho(s)=1, \rho(r)=-1$

V_3: $\rho(s)=-1, \rho(r)=-1$

V_4: $\rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\rho(r) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$|D_4| = 8 = 1 + 1 + 1 + 1 + 2^2$$

$\Rightarrow$ these are all inequivalent irreducible representations of D_4.

Conj classes: $\{1, \{r^2\}, \{r, r^3\}, \{s, sr^2\}, \{sr, sr^3\}\}$

D_4: $s^2=1, r^4=1, srs=r^{-1}$ $\quad D_4 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$[D_4, D_4] = \mathcal{L}(sr-rs,\ r-r^3,\ sr^2-s)$ is 3-dimensional

$\dim D_4/[D_4,D_4] = 5 = |\text{Conj classes in } D_4|$.

Character table:

		1	r	r^2	s	sr
	χ_0	1	1	1	1	1
	χ_1	1	1	1	-1	-1
	χ_2	1	-1	1	1	-1
	χ_3	1	-1	1	-1	1
2-dim	χ_4	2	0	-2	0	0

Dual representations.

Def. Let V be a representation of a finite group G.
Then V^* is also a representation by $V^* = Hom(V, \mathbb{C})$
$$\rho^*(g)(L) = L \circ \rho(g^{-1}), \quad L \in Hom(V, \mathbb{C})$$
Check that $\rho^*(g_1 g_2)(L) = \rho^*(g_1) \cdot \rho^*(g_2)(L)$.

Remark 1. The pairing between V and V^* is G-invariant:
$$\langle \rho^*(g) L, \rho(g) v \rangle = \langle L, \rho(g^{-1}) \rho(g) v \rangle = \langle L, v \rangle.$$

Remark 2 Finite dimensional ρ^* is irreducible $\Leftrightarrow$ ρ is irreducible

Suppose $W \subset V$ is a subrepresentation, let $W^{*\perp} = \{ \varphi \in V^* : \varphi(w) = 0 \ \forall w \in W \}$
Then $W^{*\perp} \subset V^*$ is a subrepresentation:
$$\rho^*(g)(\varphi)(w) = \varphi(\underbrace{\rho(g^{-1}) w}_{\in W}) = 0 \Rightarrow W^{*\perp} \subset V \text{ is } G\text{-subrepresentation}$$

Proposition. $\chi_{\rho^*}(g) = \overline{\chi_\rho(g)}$

Proof: Since the matrix of $\rho^*(g)$ is $(\rho(g^{-1}))^T \Rightarrow \chi_{\rho^*}(g) = \chi_\rho(g^{-1})$

$\chi_V(g) = \sum \lambda_i$ eigenvalues of $\rho(g)$ in V : $(\rho(g))^{|G|} = 1 \Rightarrow \lambda_i$ are roots of 1.

$$\Rightarrow \chi_{\rho^*}(g) = \chi_\rho(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\chi_\rho(g)}.$$

$\blacksquare$

Tensor products of representations.

Def V, W two vector spaces $\Rightarrow V \otimes W$ spanned by

$$v \otimes w \Big/ \begin{array}{l} (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \\ v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2 \\ (k \cdot v) \otimes w = v \otimes k \cdot w = k(v \otimes w) \end{array}$$

Ex. $V = \{a, b\}$ basis $\quad W = \{u, w\}$ basis

$V \otimes W = \{a \otimes u, a \otimes w, b \otimes u, b \otimes w\}$ $\qquad \dim V \otimes W = (\dim V) \cdot (\dim W)$

Def. If V, W are representations of $G \Rightarrow V \otimes W$ is a representation with $\rho_{V\otimes W}(g)(v \otimes w) = \rho_V(g) v \otimes \rho_W(g) w$

Then $\rho_{V\otimes W}(g_1 g_2) = \rho_{V\otimes W}(g_1)\,\rho_{V\otimes W}(g_2)$.

Claim 1. $\chi_{V\otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$

Matrix of $\rho_V(g) = A$, Matrix of $\rho_W(g) = B$

Matrix of $\rho_{V\otimes W}(g) = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & & \\ & & & \\ & & & \end{pmatrix}$

$\Rightarrow \operatorname{Tr}_{\rho_{V\otimes W}}(g) = \sum a_{ii} \operatorname{Tr}_{\rho_W}(g) = \operatorname{Tr}_{\rho_V}(g)\operatorname{Tr}_{\rho_W}(g) = \chi_V(g) \cdot \chi_W(g)$.

Def. If V, W are representations of G, then $\mathrm{Hom}(V,W)$ is a G-representation by
$$\rho(g): \varphi \longrightarrow \rho_W(g)\circ\varphi\circ\rho_V(g^{-1})$$

Claim 2. If V, W finite dimensional representations of G, then
$$W\otimes V^* \simeq \mathrm{Hom}(V,W) \quad \text{as a } G\text{-representation.}$$

Proof. $\{f_i\}_{i=1}^{k}$ basis in W, $\{e_j^*\}_{j=1}^{m}$ basis in V

$\Rightarrow$ $\{f_i\otimes e_j^*\} \longrightarrow f_i\, e_j^*(v) \in \mathrm{Hom}(V,W)$ elementary E_{ij} matrix

$F: W\otimes V^* \to \mathrm{Hom}(V,W) \qquad F(w\otimes L) = \varphi : v \Rightarrow w\cdot \underbrace{L(\overset{\in V}{v})}_{\in \mathbb{C}} \in W$

action on $W\otimes V^*$

$F(\rho_W(g)w,\ L\circ\rho_V(g^{-1})) = \varphi: v \to \rho_W(g)w\cdot L(\rho_V(g^{-1})v)$

$\quad\quad //$

$\underset{\text{action on } \mathrm{Hom}(V,W)}{\rho(g)}\ F(w\otimes L) = \rho_W(g)\,(w\cdot L(\rho_V(g^{-1})v))$

$\blacksquare$

Theorem. For any finite dimensional representations V, W of G

we have: $$(\chi_V, \chi_W) := \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} = \dim \operatorname{Hom}_G(V, W).$$

If V, W are both irreducibles

$$\Rightarrow \qquad (\chi_V, \chi_W) = \begin{cases} 1 & V \simeq W \\ 0, & V \not\simeq W \end{cases}.$$