Rings and modules – Final

30.01.2017, 8:15-11:15

Your Name ______________________________

This examination booklet contains 7 problems on 24 sheets of paper including the front cover and the empty sheets.

Do all of your work in this booklet, if you need extra paper, ask the proctors to give you yellow paper, show all your computations and justify/explain your answers. Calculators, books, notes, electronic devices etc. are NOT allowed.

Problem	Possible score	Your score
1	30	
2	5	
3	20	
4	10	
5	10	
6	10	
7	15	
Total	100	

By k we always denote an arbitrary field.

Question 1 [30]

For each of the following rings determine the Krull dimension, the nilradical and one minimal primary decomposition of (0).

(a) $R = k[x,y]/(x^2, y^2)$. In this case also consider the following module $M := R/(\overline{x}) \oplus R/(\overline{y})$, where the bar denotes the residue classes of the corresponding elements. Compute $\dim_k M$ and $\dim_k R$. Is M isomorphic to R? [12]

(b) $R = k[x,y]/(x^3, xy^2)$. [6]

(c) $R = k[x,y]/(x^3 + (y^2 + y)x + y)$. [6]

(d) $R = k[x,y,z]/(x^2 + y^2 + z^2)$ (your answer should depend on the characteristic of k). [6]

QUESTION 2 [5]

Let R be a (not necessarily commutative) ring. What does it mean for a module M to be simple? Prove that if $0 \neq M$ is a simple R-module then

$$End_R(M) = \{\phi : M \to M | \phi \text{ is an } R\text{–module homomorphism}\}$$

is a skew field.

Question 3 [20]

Let R be a commutative ring. We say that an R-module M is Artinian if it satisfies the descending chain condition on submodules; that is if every descending chain of submodules

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$$

satisfies $M_i = M_{i+1}$ for all i sufficiently large.

(a) Suppose M has a submodule N. Prove that M is an Artinian R-module if and only if both N and M/N are Artinian. [10]

(b) Suppose R is Artinian as an R-module. Prove that every prime ideal of R is maximal.

[Hint: reduce it to the integral domain case by quotienting, and then show that every Artinian integral domain is a field.] [10]

Question 4 [10]

Let A be the following matrix with entries in $\mathbb{Z}$:

$$A = \begin{pmatrix} -2 & 3 & -1 \\ 6 & -7 & 3 \\ 0 & 6 & 2 \end{pmatrix}$$

Put A into Smith normal form. Hence put the $\mathbb{Z}$-module M generated by m_1, m_2 and m_3 and subject to the relations

$$\begin{gathered} -2m_1 + 6m_2 = 0 \\ 3m_1 - 7m_2 + 6m_3 = 0 \\ -m_1 + 3m_2 + 2m_3 = 0 \end{gathered}$$

into the form described by the structure theorem for finitely generated modules over a PID.

Question 5 [10]

Let R be a commutative ring containing a multiplicative subset T. Prove, using the universal property of localisation, that $T^{-1}(R[x]) \cong (T^{-1}R)[x]$.

Question 6 [10]

State and prove the Noether normalisation theorem for the case of infinite base fields. You can use without proof all the preliminary lemmas and propositions we proved before the actual proof of Noether normalisation.

QUESTION 7 [15]

(a) Let R be a commutative integral domain. Prove that if R is a UFD then it is integrally closed (in $\mathrm{Frac}(R)$). [5]

(b) Let $R = k[x, y, z]/(y^3 + y^2x^2 + yx^2 + x^3z)$. You may assume without proof that R is an integral domain. Compute the integral closure of R. [10]
