Subspaces, products, quotients and disjoint unions

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Fall 2023

1 Final and initial topologies

Consider the following two motivating questions:

Question 1.1. *Given a set X and a collection of topological spaces (Y_i, τ_i) together with functions $f_i : X \to Y_i$ can we construct a topology on τ_X on the source space X such that all the functions f_i are continuous?*

Question 1.2. *Given a set Y and a collection of topological spaces (X_i, τ_i) together with functions $g_i : X_i \to Y$ can we construct a topology on τ_Y on the target space Y such that all the functions g_i are continuous?*

If we can answer Question 1.1 positively, then such a topology τ_X is called **initial** with respect to the collection of functions $f_i : X \to Y_i$. And the answer is yes! We can construct τ_X. Since continuity translates into "inverse images of open sets are open", we can consider

$$\mathcal{B} = \{f^{-1}(V_i)\,;\, V_i \in \tau_i\}$$

and we simply let τ_X be the coarsest (smallest) topology on X making all sets in $\mathcal{B}$ open. But more is true, such τ_X will be unique because an initial topology (with respect to the collection of functions $f_i : X \to Y_i$) must satisfy the following universal property:

- If (Z, τ_Z) is any topological space and $\varphi : Z \to X$ is a function, then φ is continuous if and only if $f_i \circ \varphi$ is continuous for all i.

Similarly, if we can answer Question 1.2 positively, then such a topology τ_Y is called **final** with respect to the collection of functions $g_i : X_i \to Y$. Again we can indeed construct τ_Y, we simply let

$$\tau_Y = \{V \subset Y\,;\, g^{-1}(V) \in \tau_i \text{ for all } i\}$$

which is the finest (largest) topology on Y making all the g_i continuous. And this topology will further be unique because it satisfies the following universal property:

- If (Z, τ_Z) is any topological space and $\varphi : Y \to Z$ is a function, then φ is continuous if and only if $\varphi \circ g_i$ is continuous for all i.

1.1 The subspace and the product topologies

Definition 1.3. *Let (Y, τ_Y) be a topological space and let $X \subset Y$ be a subset. The subspace topology on X is the initial topology with respect to the inclusion $\iota_X : X \to Y$.*

Why would we want to construct a topology on X so that the inclusion becomes continuous? Because we would like for the restriction of a continuous function $Y \to Z$ to X to still be a continuous function.

Definition 1.4. *Let (Y_1, τ_1) and (Y_2, τ_2) be topological spaces. Then the product topology on $X = Y_1 \times Y_2$ is the initial topology with respect to the collection $\{\pi_i : Y_1 \times Y_2 \to Y_i\}$, where each π_i denotes the usual i-th projection $(y_1, y_2) \mapsto y_i$.*

1.2 The quotient and the disjoint union topologies

Definition 1.5. *Let (X_1, τ_1) and (X_2, τ_2) be topological spaces. Then the disjoint union (or sum) topology on $Y = X_1 \sqcup X_2 = \{(x, i)\,;\, x \in X_i\}$ is the final topology with respect to the collection $\{g_i : X_i \to Y\}$, where each g_i denotes the usual inclusion $x \mapsto (x, i)$.*

Definition 1.6. *Let (X, τ) be a topological space, let Y be a set and let $\pi : X \to Y$ be a surjection. Then the quotient topology on Y determined by π is the final topology with respect to π.*

2 Some category theory

Definition 2.1. *A category $\mathcal{C}$ consists of the following data:*

- *A class of objects denoted by $\boldsymbol{Ob}(\mathcal{C})$*
- *For every two objects $X, Y \in \boldsymbol{Ob}(\mathcal{C})$, there exists a set $Hom(X, Y)$ whose elements are called morphisms from X to Y*
- *For every three objects $X, Y, Z \in \boldsymbol{Ob}(\mathcal{C})$, there exists an associative operation*

$$\circ : Hom(X, Y) \times Hom(Y, Z) \to Hom(X, Z)$$

  called composition.
- *For every object $X \in \boldsymbol{Ob}(\mathcal{C})$, there exists a morphism $id_X \in Hom(X, X)$ such that for all objects $Y \in \boldsymbol{Ob}(\mathcal{C})$ and all morphisms $f \in Hom(X, Y)$ we have $f \circ id_X = f = id_Y \circ f$.*

Example 2.2. *The category* ***Top*** *whose objects are topological spaces and the morphisms are the continuous functions.*

Definition 2.3. *Let $\mathcal{C}$ be a category and pick $X, Y \in \boldsymbol{Ob}(\mathcal{C})$. We say $f \in Hom(X, Y)$ is an isomorphism if there exists $g \in Hom(Y, X)$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.*

Thus, the isomorphisms in **Top** are the homeomorphisms.

2.1 Products and coproducts

Definition 2.4. *Let $\mathcal{C}$ be a category and let $Y_1, Y_2 \in \boldsymbol{Ob}(\mathcal{C})$. Then the product of Y_1 and Y_2 in $\mathcal{C}$, if it exists, is the object denoted by $Y_1 \times Y_2$ equipped with morphisms $\pi_i \in Hom(Y_1 \times Y_2, Y_i)$ and satisfying the following universal property:*

- *Given any object $Z \in \boldsymbol{Ob}(\mathcal{C})$ with morphisms $f_i \in Hom(Z, Y_i)$ there exists a unique morphism $f \in Hom(Z, Y_1 \times Y_2)$ which factors the f_i through the π_i.*

Thus, the cartesian product with the product topology is a (categorical) product in **Top**.

Definition 2.5. *Let $\mathcal{C}$ be a category and let $X_1, X_2 \in \boldsymbol{Ob}(\mathcal{C})$. Then the coproduct of X_1 and X_2 in $\mathcal{C}$, if it exists, is the object denoted by $X_1 \sqcup X_2$ equipped with morphisms $g_i \in Hom(X_i, X_1 \sqcup X_2)$ and satisfying the following universal property:*

- *Given any object $Z \in \boldsymbol{Ob}(\mathcal{C})$ with morphisms $f_i \in Hom(X_i, Z)$ there exists a unique morphism $f \in Hom(X_1 \sqcup X_2, Z)$ such that $f_i = f \circ g_i$*

And, similarly, the disjoint union with the disjoint union topology is a (categorical) coproduct in **Top**.

2.2 Equalizers and coequalizers

Definition 2.6. *Let $\mathcal{C}$ be a category, let $X, Y \in \boldsymbol{Ob}(\mathcal{C})$ and let $f, g \in Hom(X, Y)$. Then the equalizer of this data, if it exists, is the object $A \in \boldsymbol{Ob}(\mathcal{C})$ equipped with a morphism $a \in Hom(A, X)$ such that $f \circ a = g \circ a$ and satisfying the following universal property:*

- *Given any object $Z \in \boldsymbol{Ob}(\mathcal{C})$ and a morphism $\varphi \in Hom(Z, X)$ such that $f \circ \varphi = g \circ \varphi$ there exists a unique morphism $\tilde{\varphi} \in Hom(Z, A)$ such that $a \circ \tilde{\varphi} = \varphi$.*

In **Top** the equalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is the set $A = \{x \in X\, ;\, f(x) = g(x)\}$ equipped with the subspace topology where the morphism $a : A \to X$ is the inclusion.

In fact more is true, if (X, τ) is a topological space and $A \subset X$, then A equipped with the subspace topology and the inclusion morphism $A \to X$ is an equalizer of $X \overset{\pi}{\underset{*}{\rightrightarrows}} Y$, where:

- $Y = X/\sim$ is the quotient of X by the equivalence relation given by identifying all the points in A and we equip Y with the corresponding quotient topology
- $\pi : X \to X/\sim$ is the canonical quotient map $x \mapsto [x]$
- $* : X \to X/\sim$ is the constant map $x \mapsto [a]$, where $a \in A$

Definition 2.7. *Let $\mathcal{C}$ be a category, let $X, Y \in \boldsymbol{Ob}(\mathcal{C})$ and let $f, g \in Hom(X, Y)$. Then the coequalizer of this data, if it exists, is the object $B \in \boldsymbol{Ob}(\mathcal{C})$ equipped with a morphism $b \in Hom(Y, B)$ such that $b \circ f = b \circ g$ and satisfying the following universal property:*

- *Given any object $Z \in \boldsymbol{Ob}(\mathcal{C})$ and a morphism $\varphi \in Hom(Y, Z)$ such that $\varphi \circ f = \varphi \circ g$ there exists a unique morphism $\tilde{\varphi} \in Hom(B, Z)$ such that $\tilde{\varphi} \circ b = \varphi$.*

In **Top** the coequalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is the set $Y/\sim$, the quotient of Y by the equivalence relation on Y given by identifying all points in $f(A)$ (with A as above), equipped with the quotient topology where the morphism $b : Y \to Y/\sim$ is the corresponding quotient map.

And, again, more is true, if (Y, τ) is a topological space and $\pi : Y \to B$ is a surjection, then B equipped with the quotient topology determined by π together with the quotient morphism $\pi : Y \to B$ is a coequalizer of $Y \overset{\mathrm{id}_Y}{\underset{f}{\rightrightarrows}} Y$, where:

- f is defined by $f(y) = \beta(\pi(y))$ and $\beta : B \to Y$ is any morphism such that $\pi(\beta(\pi(y))) = \pi(y)$

Remark 2.8. *Products and equalizers are examples of (categorical) limits, whereas coproducts and coequalizers are examples of (categorical) colimits.*