

Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Hint: For exercises 1, 2, and 3, look for a solution given as a Fourier series. Using the initial equation, find the equations containing the Fourier coefficients as the unknowns. Solve these equations to get the Fourier coefficients.

Exercise 1 (Ex 17.10 page 272).

Find a 2π -periodic solution $y(x)$ for the following differential equation :

$$y'(x) + y(x) = f(x)$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ is a 2π -periodic function defined by

$$f(x) = \begin{cases} \left(x - \frac{\pi}{2}\right)^2 & \text{if } 0 \leq x < \pi, \\ \frac{\pi^2}{2} - \left(x - \frac{3\pi}{2}\right)^2 & \text{if } \pi \leq x < 2\pi. \end{cases}$$

Exercise 2 (Ex 17.8 page 271).

1. Calculate the Fourier series for the even and 2π -periodic function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \sin(3x) \quad \text{if } x \in [0, \pi].$$

2. Find a function y defined by $y(x)$, even and 2π -periodic that verifies the equation :

$$y(x) - 2y(x - \pi) = \sin(3x) \quad \text{for } x \in [0, \pi].$$

Exercise 3 (Exemple 17.6 page 268).

Find a 2π -periodic solution $y(x)$ for the following equation :

$$y(x) + 2y(x - \pi) = \cos x + 3\sin(2x) + 4\cos(5x) \quad \text{for } 0 \leq x \leq 2\pi.$$

Hint: For all $n \in \mathbb{N}$, we have the trigonometric formulas

$$\cos(nt - n\pi) = (-1)^n \cos(nt), \quad \sin(nt - n\pi) = (-1)^n \sin(nt).$$

Exercise 4 (Ex 15.1 page 239).

Compute the Fourier transform for the function f defined by :

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-x} & \text{if } x > 0. \end{cases}$$

Exercise 5.

Compute the Fourier transform of the function f defined by :

$$f : \mathbb{R} \rightarrow \mathbb{R}; x \mapsto f(x) = \begin{cases} 1 & \text{if } x \in [-1, 1[, \\ 0 & \text{else.} \end{cases}$$

Exercise 6.

Draw the graph for the following functions and find their Fourier transforms :

$$\begin{aligned} \text{a) } f(x) &= \begin{cases} \sqrt{\frac{\pi}{2}}, & \text{if } -1 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases} \\ \text{b) } g(x) &= \begin{cases} \pi + \frac{\pi}{2}x, & \text{if } -2 \leq x \leq 0, \\ \pi - \frac{\pi}{2}x, & \text{if } 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Which is the link between the functions f et g ?

Exercise 7 (Ex 15.2 page 239).

We consider the function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by $f(x) = e^{-x} \cos x$.

- a) Compute the Fourier cosine transform of the even extension to $\mathbb{R}$ of f .
- b) Compute the Fourier sine transform of the odd extension to $\mathbb{R}$ of f .