Exercise 13: Iterated maps and the period doubling route to chaos: the sine map

December, 10th 2024

Course: Dynamical systems in biology (BIO 341)

Professors: Julian Shillcock & Felix Naef

SSV, BA5, 2023

1.0 The sine map

The sine map is defined by the iteration scheme:

$X_{n+1} = r \sin(\pi x_n)$

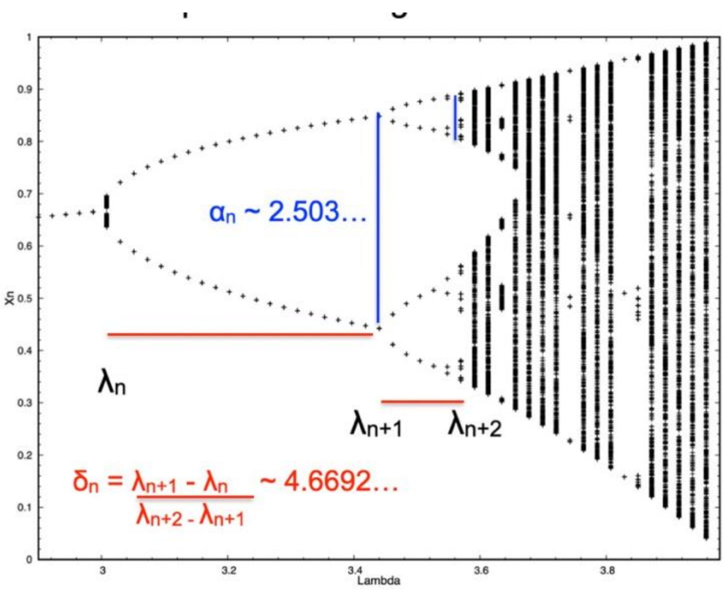
For a constant parameter $0 < r < 1$, and an initial value that satisfies $0 < x_0 < 1$.

1.1 Write a python code that iterates the above map, given the three parameters: r, the number of iterations N, and an initial point x_0. Either write the iterates to a file for later plotting, or plot them directly in your python code. Hint. Use 2000 iterations and discard the first 1000 points before using the data for plotting or calculations to remove the initial transients in the map.

Wrap the iteration function in a loop over a user-defined range of r values, and plot all the iterates for each r value in the same graph. It should start to look like the figure below, i.e., for each value of r, all the iterates produced from the initial value x_0 should be plotted vertically. If you want, you can change the value of x0 for each value of r, but because the map is "chaotic", it doesn't really matter. NB. Don't use x_0 = 0, or 1!

1.2 For r = 0.6, iterate the map for at least 5 initial values randomly chosen between (0, 1). What do you observe? Then repeat for r = 0.72 and 0.75. What do you observe in your plot of x_n against increasing r values?

1.3 Keep increasing r, and plot the iterates produced. At what r value does "chaos" set in for the sine map? i.e., when does a (nearly) infinite number of fixed points appears (cp. Logistic map shown in the figure below, for which lambda ~ 3.6 is the transition to chaos. Note that lambda in the logistic map plays the role of r for the sine map.) Do you observe any "stability windows" in which the number of fixed points is small? (cp. Logistic map for lambda ~ 3.82, where only 3 fixed points appear.)



2.0 Feigenbaum numbers

2.1 Extend your code to output the iterates from the sine map for a sequence of r values between 0.1 and 1, taking at least 20 values, and randomly setting the initial point for each one. Plot the fixed point(s) on the Y axis against the r value on the X axis, so you get a plot similar to the one above.

Then repeat this accurately enough that you can estimate the successive r values at which the number of fixed points doubles (you may need more than 20 values of r, and you may need to zoom in on small portions of the r-axis to get sufficient accuracy. So you'll need to examine many little graphs across the r axis to locate the values of r where the period doubling occurs.

2.2 From your graphs in 2.1, estimate the Feigenbaum numbers δ and α, which are defined in the figure as the ratio of successive changes in the r value and successive splitting of the fixed points when a new fixed point(s) appears.

How many successive splittings can you measure accurately enough to get values for the Feigenbaum numbers?

2.2 Say how your estimates of the Feigenbaum numbers for the sine map compare with those for the logistic map shown in the figure.

3.0 Fractal dimension of the sine map

3.1 Find a value of r such that you get a lot of fixed points (i.e., you are in a region for the sine map corresponding to the region r ~ 3.95 for the logistic map). Generate 10^5 points from a randomly-chosen initial point x_0, and write them to a file. Discard the first 50,000.

Duplicate the data into a second column shifting each value by one. So, for each row in the file, column one contains x_n, and column two contains x_{n+1}. Plot x_{n+1} against x_n.

3.2 For the 50,000 points from 3.1, and for a series of values of ε = 1, 0.5 0.25, $1/2^n$… create a histogram of the number of points within a distance ε of x_i for a sequence of values of x_i ranging from 0 to 1. Then average the number of points in all the bins for each x_i value, to get an average $<N(\varepsilon)>$ for the number of points that lie in bins of size ε across the set of points. Repeat this for each ε.

Plot ln($<N(\varepsilon)>$) against ln(ε) and measure the slope to obtain the "fractal dimension" of the fixed points of the sine map for a single r value.

What is the distribution of fixed points for this value of r?

4.0 Universality in chaos

4.1) Add a new function to your code from Section 1.0, that iterates the logistic map:

$x_{n+1} = \lambda\, x_n\, (1 - x_n)$

where λ is in the range (0, 4), and x_0 is in the range (0, 1). Note that I have used r and λ because they have different allowed ranges.

a) Choose values of r and λ near the beginning of their range (but not zero), and plot the first iterate obtained from both maps on the same graph, i.e., if f(x) is the logistic map and g(x) is the sine map, plot $f(x_0)$ and $g(x_0)$ for many x_0 between 0 and 1. What do you observe? Can you find values of r, λ so that the curves nearly overlap?

b) What property of the map functions do you think is necessary for two discrete maps to have similar long-time behaviour (referred to in the literature as being in the same *universality class*.)

4.2) Comment on how your bifurcation curve for the sine map compares to that obtained from the logistic map shown above.

4.3) What implications does this have for using simple maps like the logistic map for making predictions about complex natural system such as turbulent flow of fluids?